A Guide to the study of ***Functions*** at Post –Primary school

Children begin to develop the function concept in early childhood when they observe and continue patterns of objects in everyday life. They continue the preparation for functions in primary school by exploring patterns in numbers and looking for regularities. It is, however at ***JCOL*** when they are formally introduced to the notion of a function. They learn about *the meaning and notation associated with functions* and

- engage with the concept of a function, domain , co-domain and range
- make use of function notation f(x)= ;   f:x $\rightarrow$, and y=

At this level students make connections between their study of relationships in *strand* ***4 algebra sections 4.1 to 4.5***, sets in ***strand 3 number section 3.5*** and data types in ***strand 1 section 1.4.*** They learn that ***function*** is a mathematical term that refers to a ***certain kind*** of relationship between two sets. One set called the ***domain*** and the other the ***range***. A function is a correspondence between these sets; from each element in the ***domain*** to exactly one element in the ***range.***

At this level students understand the ***domain*** as the set of what they have hitherto referred to as ***inputs*** or ***what goes into the function***, and the set of ***outputs*** as the ***range*** or what ***actually*** comes out of the function. They identify another set, the ***co-domain*** as the set of what ***possibly*** comes out of the function and view ***the range*** as a subset of this set. Having initially engaged in ***sections 4.1-4.5*** with a variety of relationships derived from familiar, everyday experiences the more formal definitions are gradually introduced. When examining a ***money box*** situation, for example, in which a box has €5 to start with and €2 added every day students at this level students would identify from the situation the ***domain*** as the set of whole numbers, the ***co-domain*** as the set of whole numbers or maybe even the set of whole numbers greater than or equal to 5 and the ***range*** as the set of odd whole numbers 5,7,9…… In addition they would be able to identify the ***domain*** and ***range*** in each of the other representations; tabular, ordered pair, and graphical.

The exploration of the ***co-domain*** and decision making around outcomes that are possible and those that are not presents an ideal opportunity to make connections with data types from ***section 1.4*** and to reinforce the concept of ***discrete*** and ***continuous*** data. By considering the possible outcomes of a function and how they should be represented graphically the students are presented with ***discrete*** and ***continuous*** data in a context other than statistics and can begin to categorise situations that produce each type of data.

At this level students use function notation as *shorthand*, for describing the correspondence in terms of input and output. Initially the notation is used in conjunction with the situation and students should recognise that the correspondence is built into the notation. In the moneybox situation described above ***f*****(2)** should be interpreted as

the amount of money in the money box on day 2 and ***f*(x)** as the amount of money in the money box on any given day. They should recognise the **x** as a place holder and realise that ***f(b)*** would describe exactly the same situation. The students' understanding of the notation should be explored in relation to the context so that, for example, they understand the difference between ***f(x+2***), ***f(x)+2*** and ***2 f(x) .*** They should be able to explain that ***f(x+2***) is the amount of money in the money box 2 days after any given day; ***f(x)+2*** is the amount of money in the money box on any given day plus €2, and ***2 f(x)*** is twice the amount of money on any given day.
Later, students consolidate their learning from ***sections 4.3 ,5.1 and 5.2*** as they work to develop fluency in moving between the different representations (notation, graphical and the context), and use this fluency to solve purely mathematical problems or those set in a context. ***JCOL*** students explore the notation with linear functions and quadratic functions with whole number coefficients whilst at ***JCHL*** students extend their exploration of function notation to include quadratic functions with integer coefficients and simple exponential functions. At this level students should understand that a function can also be described in terms of its behaviour, for example, *over what input values is it increasing, decreasing or constant? For what input values are the output values positive, negative or zero?* This focus on function behaviour offers an ideal opportunity to reinforce the concept of domain and appreciate the usefulness of the graphical representation, since these behaviours are easily ***seen*** in a graph.
A discussion of function behaviour offers an ideal segue to ***LC*** ; by this level students should be developing ways of thinking that are general and which allow them to approach any type of function, work with it, and understand how it behaves, rather than regarding each function as a completely different ***thing*** to study. With a basis of experiences in building specific functions from scratch, beginning with an exploration of relations in ***sections 4.1- 4.5*** and progressing to generalisation, first in words – ***amount of money in the box = 2 (day number) + 5*** – and later using algebraic notation – ***y* = 2*x*+5** – a well-developed concept of equality allows students to make sense of the notation ***f*(*x*) = 2*x*+5** , interpreting it as ***the output is 2x +5*** when ***the input is x***. In addition, they develop their understanding of the equivalence of ***y*** and ***f(x),*** not only in this algebraic representation but also in the tabular and graphical representations. Now students should start to develop a notion of naturally occurring families of functions that deserve particular attention. For example, they should see linear and exponential functions as arising out of growth principles. Similarly, they should see quadratic, polynomial, and rational functions as belonging to the one system. Developing this notion takes time and students can start getting a feel for the effects of different parameters by playing around with the effect of the input and output variables on the graph of simple algebraic transformations. Quadratic (***LCOL***) and absolute value

functions (***LCHL***) are good contexts for getting a sense of the effects of many of these transformations.

Proficient mathematicians will make use of structure to help solve problems and at all levels students should be encouraged to look for and make use of structure. Consequently, students should develop the practice of writing expressions for functions in ways that reveal the key features of the function. At ***LCHL***, exploring quadratic functions provides an ideal opportunity for developing this ability, since the three principal representations for a quadratic expression – expanded, factored, and completed square – each give insight into different aspects of the function.

At ***LCHL***, students extend the idea of the co-domain introduced at ***JCOL*** when they begin to categorise functions as ***surjective***, ***injective*** or ***bijective***. Exploring the effects of limiting the domain and co-domain on the function 'status' reinforces the difference between them and also helps students to make sense of the categorisation.

At ***LC*** by examining contexts where change occurs at discrete intervals (such as payments of interest on a bank balance) or where the input variable is a whole number ***section 3.1*** they come to recognise that a ***sequence*** is a function whose domain is a subset of the set of integers. For example, when considering the sequence 5, 8, 11,14 by choosing an ***index*** that indicates which term they are talking about and which serves as the input value to the function, a student could make a table showing the correspondence and describe the sequence using function notation $f(x) = 3x + 2, x \geq 1$ with the domain included in the description. Students are faced once again with the concept of ***discrete*** and ***continuous*** data when they attempt to represent a sequence graphically.

LCHL students begin engaging with the concept of ***the inverse*** of a function by first getting to grips with the idea of ***going backwards*** from output to input. They can get this sense of determining the input when the output is known by using a table or a graph of the function under examination. To reinforce this idea, correspondences between equations giving specific values of the functions, table entries, and points on the graph can be noted. Eventually students need to generalise the process for finding the input given a particular output and are required to generalise the process for ***bijective*** functions only. A well-developed concept of notation and equality is required if students are to make sense of the generalisation.

To help students develop the concept that "inverse" is a relationship between two functions rather than a new type of function the classroom focus should be on "inverses of functions". Questions such as, "*What is the inverse of this function*?" and "*Does this function have an inverse?*" are useful in keeping the focus on the relationship idea.

Connections can be made with the notion of ***function composition*** by examining the relationship between the composition of f^{1} with ***f.*** This relationship, the ***identity function***, which assigns each function to itself allows students to deepen their understanding of inverses in general since it behaves with respect to composition of functions the way the multiplicative identity, 1, behaves with multiplication of real numbers. Now students can verify by composition (in both directions) that given functions are inverses of each other. They can also refine their informal "going backwards" idea, as they consider inverses of functions given by graphs or tables. They get a sense that "going backwards" interchanges the input and output and therefore the stereotypical roles of the letters ***x*** and ***y*** and can reason why the graph of $y=f^{1}(x)$ will be the reflection across the line ***y=x*** of the graph ***y=f(x).***

Section 3.2 provides ***LCHL*** students with further opportunity to reinforce the concept that "inverse" is a relationship between two functions, here students are required to not only understand logarithms as functions but also as inverses of exponential functions. Students can think of the logarithms as unknown exponents in expressions with base 10 and use the properties of exponents when explaining logarithmic identities and the laws of logarithms.

LC section 5.2 introduces students to the concept of a ***limit,*** a powerful tool for them as they start bringing together their ability to use graphs to reason about rates of change ***(JC sections4.1-4.5)*** and start thinking about the slope of a tangent line to a curve. Students can develop their understanding of differentiation and ***why*** the rules work by examining differentiation of linear and quadratic functions from fist principles although students at ***LCHL*** only will be examined in this process.

At ***LCHL*** students build on their ability to approximate area in ***section 3.4*** by investigating the area under a function. Starting by finding the area between a given linear function and the x-axis and progressing to *finding* the upper boundary function themselves they come to in ***section 5.2*** to recognise integration as the reverse process of differentiation***.*** By examining the problem of finding the average value of a function over a given interval they progress to a deeper understanding of the process of integration. The ability to determine areas of plane regions bounded by polynomial and exponential curves eases the transition from computing discrete probabilities to continuous ones ***section 1.3***. By understanding the ***Normal distribution*** as a ***probability density function*** students can understand why it is used to find probabilities for continuous random variables.