The Derivative

Making sense of Differentiation

Investigating the problem of *speed* can lead to interesting understandings in mathematics.

How can you measure the speed of a moving object at a given instant in time? Or indeed

What is meant by the term *speed?*

Defining speed has wide-ranging implications – not just for solving speed problems but for measuring the rate of change of any quantity.

Your investigative journey will lead you to the key concept of *derivative*, which forms the basis of your study of calculus

How do you measure speed?

The speed of an object at an instant in time is surprisingly difficult to define precisely. Consider the statement. "*At the instant he crossed the finish line in 2009 Usain Bolt was travelling at 28 mph*". How can such a claim be substantiated? A photograph taken at that instant would be no help at all as it would show Usain Bolt motionless. There is some paradox in trying to study Usain's motion at a particular instant in time since, to focus on a single instant, you effectively stop the motion!

Problems of motion were of central concern to Zeno and other philosophers as early as the 5^{th} century B.C. The modern approach, made famous by Newton's calculus, is to stop looking for a simple notion of speed at an instant, and instead to look at speed over small time intervals containing the instant. This method sidesteps the philosophical problems mentioned earlier but introduces new ones of its own.

Gedankenexperiment

The ideas mentioned above can be illustrated in an idealised thought experiment [Gedankenexperiment] that assumes we can measure distance and time as accurately as we want.

Think about the speed of a piece of plasticine that is thrown straight upward into the air at

$t = 0$ seconds. The plasticine leaves your hand at high speed, slows down until it reaches its maximum height, and then speeds up in the downward direction and finally hits the ground.

Suppose that you want to determine the speed, say, at $t = 1$ second. The table shows the height, s, of the plasticine above the ground as a function of time.

T (sec)	0	1	2	3	4	5	6
s (m)	2	27	43	49	45	32	9

During the 1st second the plasticine travels 27 – 2 = 25m, and during the second second it travels only 43 – 27 = 16m. Hence, the plasticine travelled faster over the 1st interval, $0 \le t \le 1$, than over the second interval, $1 \le t \le 2$.

Speed v Velocity

Physicists distinguish between velocity and speed. Suppose an object moves along a line. They pick one direction to be positive and say that the *velocity* is positive if it is in this direction, and negative if it is in the opposite direction. For the plasticine, upward is positive and downward is negative. Because *speed* is the magnitude or size of the velocity, it is always positive or zero.

Now … a definition

If *s(t)* is the position of an object at time *t*, then the **average velocity** over the interval $a \le t \le b$ is

$$\textbf{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(b)-s(a)}{b-a}$$

In words, the **average velocity** of an object over an interval of time is the net change in position of the object during this interval divided by the change in time (i.e. the time interval).

Test yourself:

Use the table above to calculate the average velocity of the plasticine over the interval $4 \le t \le 5$. What is the significance of the sign of your answer?

Calculate the average velocity of the plasticine over the interval $1 \le t \le 3$.

Why is the average velocity a useful concept? Well, it gives a rough idea of the behaviour of the plasticine. If two pieces of plasticine are thrown into the air, and one has an average velocity of 10 m/s over the interval $0 \le t \le 1$ while the second has an average velocity of 20 m/s over the same interval, the second one is moving faster.

However, average velocity over an interval does not solve the problem of measuring the velocity of the plasticine at *exactly* $t = 1$ second. To get closer to an answer to that question, you have to look at what happens near $t = 1$ in more detail. You must look at the average velocity over smaller intervals on either side of $t = 1$.

The value of the average velocity before t = 1 is slightly more than the average velocity after t = 1 but, as the size of the interval shrinks, the values of the average velocity before I = 1 and the average velocity after t = 1 get closer together. Eventually, in the smallest interval, the two average velocities are the same; this is what we define the ***instantaneous velocity*** at t = 1 to be.

More generally, you can use the same method as for $t = 1$ to find the ***instantaneous velocity*** at any point $t = a$:

on small intervals of size h around $t = a$

Average velocity $= \frac{s(a+h)-s(a)}{h}$

Try to make sense of this definition.

The ***instantaneous velocity*** is the number that the average velocities approach as the intervals decrease in size, that is, as h becomes smaller.

So you can formally define ***instantaneous velocity*** at $t = a$ to be

$$\lim_{h \to 0} \frac{s(a+h)-s(a)}{h}$$

In words, the **instantaneous velocity** of an object at time $t = a$ is given by the limit of the average velocity over an interval of time, as the interval shrinks around a.

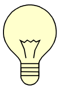
In a time of t seconds, a particle moves a distance of s metres from its starting point, where

$\boldsymbol{s} = 3\boldsymbol{t}^2$.

(a) Find the average velocity between $\boldsymbol{t} = 1$ and $\boldsymbol{t} = 1 + \boldsymbol{h}$ if:

(i) $\boldsymbol{h}$ = 0:1, (ii) $\boldsymbol{h}$ = 0:01, (iii) $\boldsymbol{h}$ = 0:001.

(b) Use your answers to part (a) to estimate the instantaneous velocity of the particle at time $\boldsymbol{t} = 1$.



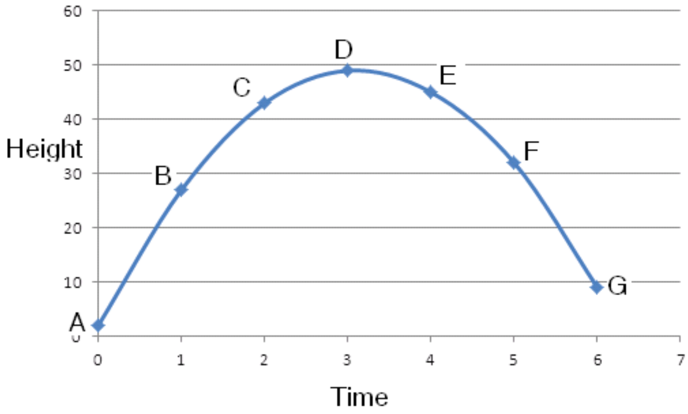
Now think …..How would this question change if $\boldsymbol{s}$ were a more complex function?

Say, for example, $\boldsymbol{s} = 3\boldsymbol{t}^2 + 4$ or $\boldsymbol{s} = \text{Sin}\ (2\mathbf{t})$

Try it out for yourself. 

Now, back to the plasticine…

The graph shows the height of the plasticine plotted against time.



Try to visualise the average velocity on this graph.

Hint: Suppose ***y*** = **s***(t)* and consider the interval $1 \le t \le 2$

Look back at the definition of average velocity…

$$\textbf{Average velocity} = \frac{\text{Change in position}}{\text{Change in time}} = \frac{s(b)-s(a)}{b-a}$$

In this situation $b = 2$ and $a = 1$

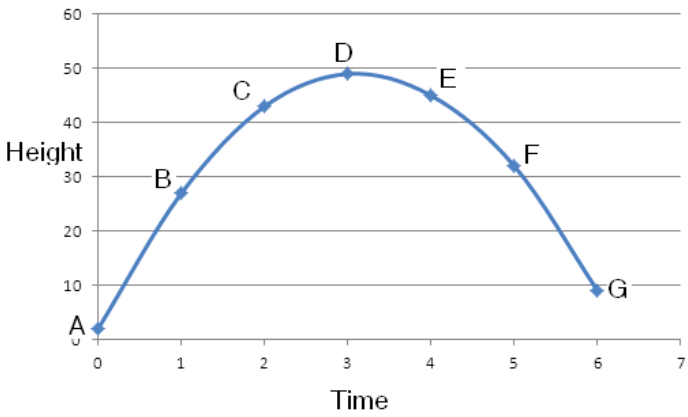
Now look at the *change in position*, $s(2)$-$s(1)$ on the graph .Mark it with a line.

Now look at the change in time, 2-1 on the graph .Mark it with a line.

Now can you make sense of this textbook definition?

The **average velocity** over any time interval $a \le t \le b$ is the slope of the line joining the points on the graph of s(t) which correspond to t = a and t =b

Use this definition to describe the velocity of the plasticine throughout its flight [i.e. is it increasing? Decreasing? Staying the same?] You may like to draw lines on the diagram.


Height
Time
A
B
C
D
E
F
G
0
10
20
30
40
50
60
1
2
3
4
5
6
7

Generalising your observations

When thinking about the plasticine you derived an expression for the **average velocity** or the "change in height divided by the change in time". Look back at your diagram and see how these two things are the same.

So now you have derived an expression for the average rate of change of height with respect to time

Average rate of change of height with respect to time $= \frac{s(a+h)-s(a)}{h}$

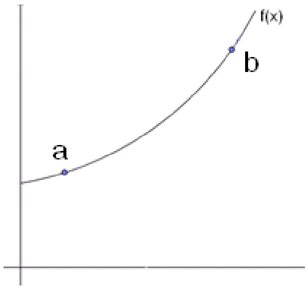
Spend some time thinking about this. Look back at your graphs; where is ***s(a+h) –s(a)*** on your graph? Where is ***h*** on your graph?

This ratio is called the ***difference quotient***. Now, apply the same analysis to any function ***f***, not necessarily a function of time:

Average rate of change of ***f*** over the interval from ***a*** to ***a+h*** $= \frac{f(a+h)-f(a)}{h}$

Time

- Make sense of this definition
- Visualise the definition on the graph below by considering how the function changes from point a to point b, a horizontal distance ***h*** away from a.



Draw a line to represent the numerator, i.e. the distance ***f*** (***a***+***h***) –***f*** (***a***)

Draw a line to represent denominator, i.e. the distance ***h***

Draw the line whose slope is $\frac{numerator}{denominator}$, i.e. ($\frac{\boldsymbol{f}\ (\mathrm{a+h})\ -\boldsymbol{f}\ (\mathrm{a})}{\boldsymbol{h}}$)

Now think about what happens as **h** becomes increasingly smaller.

What happens to this line with slope $\frac{\boldsymbol{f}\ (\mathrm{a+h})\ -\boldsymbol{f}\ (\mathrm{a})}{\boldsymbol{h}}$?

What happens to the slope of this line as **h** becomes increasingly smaller?

In other words, what is $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$?

Make sense of the following definition

The instantaneous rate of change of a function at a point *a* is defined as the derivative of the function and is written ***f'(a)***
$$f'(a) \ = \ \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
If the limit exists, then the function ***f*** is said to be ***differentiable*** at *a*.

Rate of change in context

Have you ever noticed that, when you are blowing up a balloon, it seems to blow up faster at the start and then slow down as you blow more air into it?

You can explain this mathematically by examining the rate of change of the radius with respect to the volume.

The volume V of a sphere is given by $\boldsymbol{V} = \frac{4}{3}\pi r^3$

Rearranging gives $\boldsymbol{r} = \sqrt[3]{\frac{3V}{4\pi}}$

Examine the average rate of change of the radius with respect to ***V*** over the intervals $\boldsymbol{0.5 \leq V \leq 1}$ and $\boldsymbol{1 \leq V \leq 1.5}$ to see what happens to the rate of change of the radius as the volume increases.

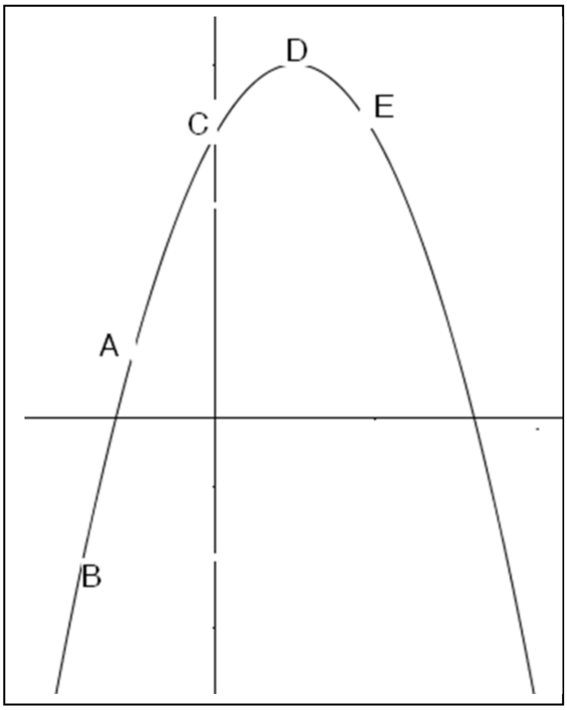
It should start to decrease, thus explaining the observation that when you blow up a balloon it seems to blow up faster at the start and then appears to slow down as you blow more air into it

Try it out for yourself

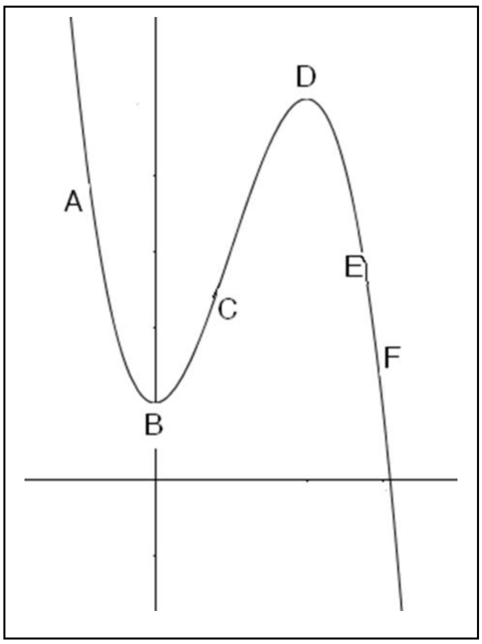
Tasks

Match the points labelled on the given curves with the slopes in the table.

Slope	Point
–4	
4	
0	
10	
8	



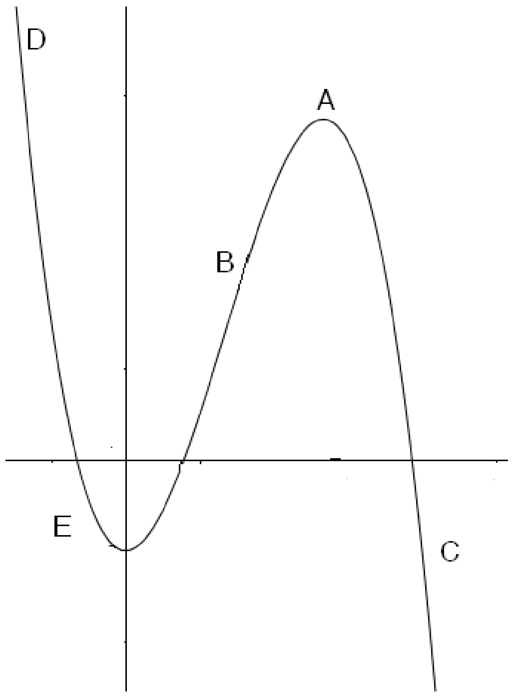
Slope	Point
–9	
0	
–10	
3	
–3.75	



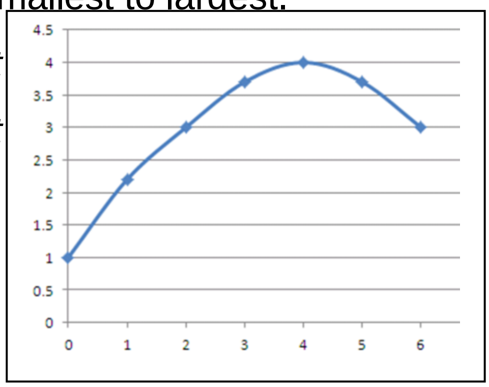
For the function shown,

(a) At what labelled points is the slope of the graph (i) positive (ii) negative?

(b) At which labelled point does the graph have (i) the greatest slope (ii) the least slope?



The graph of ***f***(t) in the diagram below gives the position of a particle at time t. List the following quantities in order, smallest to largest.

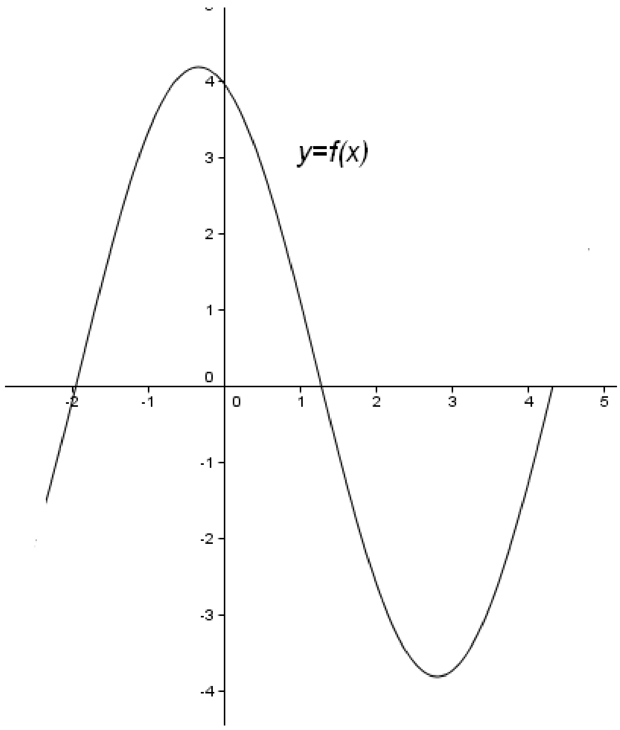


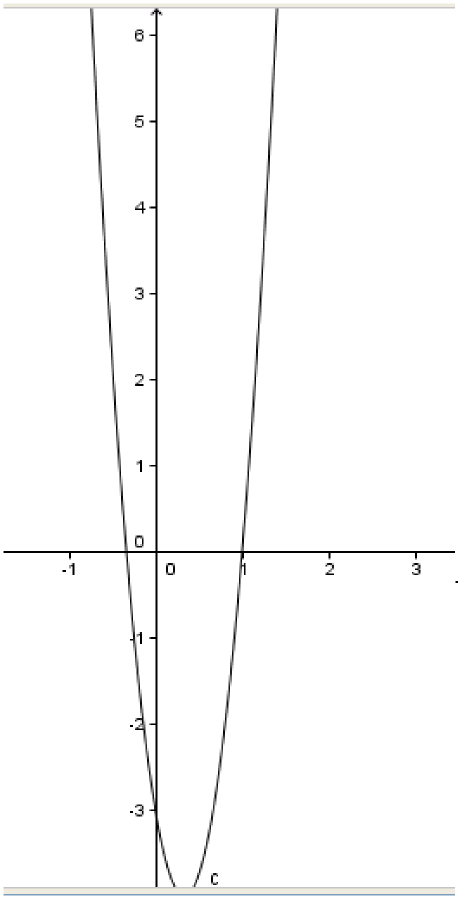
- A, average velocity between t = 1 and t
- B, average velocity between t = 5 and t
- C, instantaneous velocity at t = 1,
- D, instantaneous velocity at t = 3,
- E, instantaneous velocity at t = 5,
- F, instantaneous velocity at t = 6.

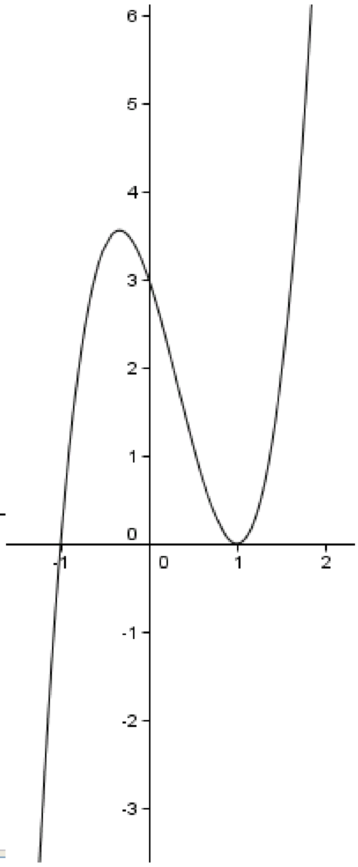
An object moves at varying velocity along a line and ***s*** = ***f***(*t*) represents the particle's distance from a point as a function of time, *t*. Sketch a possible graph for ***f*** if the average velocity of the particle between ***t*** = 2 and ***t*** = 6 is the same as the instantaneous velocity at

t = 5.

Estimate the derivative of the function ***f***(x) shown in each graph below at *x* = –2, –1, 0, 1, 2, 3, 4







The table gives values of ***c***(***t***), the concentration (μg/cm^3) of a drug in the bloodstream, at time ***t*** (min). Construct a table of estimated values for ***c'(t***), the rate of change of ***c(t)*** with respect to time

t (min)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
c(***t***)(μg/cm^3)	0.84	0.89	0.94	0.98	1.00	1.00	0.97	0.90	0.79	0.63	0.41

Compare, Examine, Discuss and Evaluate

$$C'(t) = \frac{C(t+h) - C(t)}{h}$$

$$C'(0) = \frac{C(0.1) - C(0)}{.1} = \frac{.89 - .84}{.1} = 0.5 \text{ mg/cm}^3$$

$$C'(.1) = \frac{C(0.2) - C(0.1)}{.1} = \frac{0.94 - 0.89}{.1} = 0.5 \text{ mg/cm}^3$$

$$C'(.2) = \frac{C(0.3) - C(0.2)}{.1} = \frac{0.98 - 0.94}{.1} = 0.4 \text{ mg/cm}^3$$

$$C'(.3) = \frac{C(0.4) - C(0.3)}{.1} = \frac{1.00 - 0.98}{.1} = 0.2 \text{ mg/cm}^3$$

$$C'(.4) = \frac{C(0.5) - C(0.4)}{.1} = \frac{1.00 - 1.00}{.1} = 0 \text{ mg/cm}^3$$

$$C'(.5) = \frac{C(0.6) - C(0.5)}{.1} = \frac{.97 - 1.00}{.1} = -0.3 \text{ mg/cm}^3$$

$$C'(.6) = \frac{C(0.7) - C(0.6)}{.1} = \frac{.9 - .97}{.1} = -0.7 \text{ mg/cm}^3$$

$$C'(.7) = \frac{C(0.8) - C(0.7)}{.1} = \frac{.79 - .9}{.1} = -1.1 \text{ mg/cm}^3$$

$$C'(.8) = \frac{C(0.9) - C(0.8)}{.1} = \frac{.63 - .79}{.1} = -1.6 \text{ mg/cm}^3$$

$$C'(.9) = \frac{C(1.0) - C(0.9)}{.1} = \frac{.41 - .63}{.1} = -2.2 \text{ mg/cm}^3$$