Primary

Conference

Early mathematics is surprisingly important and cognitively fundamental

NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Early mathematics is surprisingly important and cognitively fundamental

Conference: Dublin Castle 24.11.14

A conference on developing mathematical ideas with children (3-8 years) took place on Monday, November 24th in the Conference Centre in Dublin Castle. The conference explored what good mathematical experiences for children looks like in the initial years of primary school and why these experiences are important.

Delegates heard keynotes from Professor Douglas Clements (University of Denver) and Professor Elizabeth Wood (Sheffield University) and took part in workshops spotlighting innovative maths work in schools and early years services. The conference closed with the Minister for Education and Skills, Jan O'Sullivan, TD launching two research reports.

Video

Watch the introductory remarks and keynote presentations.

Dr. Sarah FitzPatrick, Deputy CEO

Watch the Introductory Remarks

Read a transcript.

Professor Douglas Clements, University of Denver

The Building Blocks of Early Mathematics

Professor Doug Clements is a Kennedy Endowed Chair in Early Childhood Learning, a Professor, and the Executive Director of the Marsico Institute of Early Learning and Literacy at the University of Denver's Morgridge College of Education.

Previously a kindergarten teacher for five years and a preschool teacher for one year, Doug has since conducted research and has been published widely in the areas of:

- The learning and teaching of early mathematics
- Computer applications in mathematics education
- Creating, using, and evaluating a research-based curriculum and in taking successful curricula to scale using technologies and learning trajectories.
- Development and evaluation of innovative assessments of mathematics achievement, as well as mathematics teaching.

Watch Professor Clemens lecture

View the slides

Video

Professor Elizabeth Wood, Sheffield University

Progression in Play and Playful Learning

Professor Elizabeth Wood teaches on the Masters and Doctoral programmes in Sheffield University in the fields of early childhood and primary education and teacher's thinking and classroom practice; policy analysis; equity issues. Elizabeth has worked on professional development action research projects with teachers in early childhood settings, primary and secondary schools.

Elizabeth's fields of research and teaching include early childhood and primary education, focusing on the following themes: learning, pedagogy and curriculum; play and learning; policy analysis and critique (national and international); equity and diversity; teachers' beliefs and practices; professionalism and critical perspectives in education.

Within the theme of play, her research focuses on pedagogy and practice, the ways in which play has been captured in policy sites, and the construction of 'educational play'. Elizabeth is interested in respectful and ethical ways of researching and understanding play from children's perspectives.

Watch Professors Woods lecture: Progression in Play and Playful Learning.

Maths in action

Watch a video on maths in action.

Maths through play

Watch a video on learning maths through play.

Conference materials

Programme	Download the programme (PDF)
Keynote Speakers	Prof Douglas Clements, University of Denver - Prof Elizabeth Wood, Sheffield University
Workshops	Check out materials from the workshop
Agenda	Conference Agenda
Research Reports	Two new maths research reports were published on 24th November 2014.

Workshops

Full list of workshops

Check out materials from the workshop below or read the full list above workshops above.

‘Equal’ sign

Facilitating young children’s understanding of the ‘equal’ sign - Dr Thérèse Dooley, St Patrick’s College, Drumcondra and Aisling Kirwan, Holy Family National School, Rathcoole, Co Dublin

3-4 years

Mathematics for children aged 3-4 years - Dr Josephine Bleach, Early Learning Initiative, National College of Ireland, Mark Shinnick, Holy Child Preschool, Rutland St and Aisling Rourke, St Andrew’s Resource Centre

3 years and under

Mathematics for children aged 3 years and under - Dr Josephine Bleach, Early Learning Initiative, National College of Ireland, Michelle Moore, Parent Child Home Programme and Moira Ward, St Andrew’s Resource Centre, Pearse St

Communicating Maths thinking and understanding

Young people communicating mathematical thinking and understanding. Ross Ó Corráin & Dr. Liz Dunphy

Supporting children at risk of experiencing difficulties in early mathematics

Evidence based approaches and conceptual frameworks for early intervention and differentiation in mathematics. Joe Travers and Orla McKeirnan

Review and Research

Research Publications

2016	Background Paper and Brief for the Development of the New Primary Maths Curriculum
2014	Mathematics in Early Childhood and Primary Education (3-8 years): Definitions, Theories, Development and Progression. NCCA Research Report No. 17
2014	Mathematics in Early Childhood and Primary Education (3-8 years): Teaching and Learning. NCCA Research Report No. 18
2014	Audit of Mathematics Curriculum Policy across 12 Jurisdictions: Commissioned Report
2014	Mathematics in Early Childhood and Primary Education (3-8 years): Executive Summaries

Review and Research

Video Reviews

Dr Gerry Sheil

Dr Gerry Shiel provides an overview of NCCA Research Reports Nos. 17 and 18, Mathematics in Early Childhood and Primary Education (3-8 years).

Dr Elizabeth Dunphy

Dr Elizabeth Dunphy highlights key messages from NCCA Research Report No. 17, Mathematics in Early Childhood and Primary Education (3-8 years): Definitions, Theories, Development and Progression.

Dr Thérèse Dooley

Dr Thérèse Dooley highlights key messages from NCCA Research Report No. 18, Mathematics in Early Childhood and Primary Education (3-8 years):Teaching and Learning.

[home]

PROGRAMME

Early mathematics is surprisingly important and cognitively fundamental

Developing mathematical ideas with children (3-8 years)

Dublin Castle Conference Centre, 24th November 2014

ncca.ie/mathsconf #maths3to8

NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

primary developments
foráis sa bhunscolaíocht

Aistear
Creatchuraclam na Luath-Óige
The Early Childhood Curriculum Framework

AGENDA

Mathematics is a way of thinking about and seeing the world. It is part of the DNA of children's conversations, their play, their daily routines and activities and their interactions with each other. Building on this, the conference aims to explore what good mathematical experiences for children look like in the initial years of primary school and why these experiences are important. The conference will showcase examples of innovative maths teaching and learning from classrooms and services around the country. The conference will conclude with the launch of two exciting research reports which provide new ideas for redeveloping the primary school mathematics curriculum beginning with junior infants to 2nd class.

9.00 – 9.15	**Check-in and refreshments**
9.45-10.00	**Welcome and introduction** Dr Sarah Fitzpatrick, Deputy Chief Executive, NCCA *Main Conference Hall*
10.00 – 11.00	**Keynote 1: The Building Blocks of Early Mathematics** Professor Douglas Clements, University of Denver *Main Conference Hall*
11.00 – 11.30	**Tea/coffee break - Foyer**
11.30-12.40	**Workshops** See workshop lists and signs for room details
12.40-1.40	**Lunch** *Castle Hall*
1.40-2.40	**Keynote 2: Progression in Play and Playful Learning** Professor Elizabeth Wood, Sheffield University *Main Conference Hall*
2.40-3.45	**Workshops** *See workshop lists and signs for room details*
3.45-4.15	**Launch of NCCA research reports:** **Mathematics in early childhood and primary education (3-8 years)** Introduction: Brigid McManus, Chairperson, NCCA Researchers: Drs Elizabeth Dunphy, Thérèse Dooley and Gerry Shiel Launch of reports by the Minister for Education and Skills, Jan O'Sullivan, TD *Main Conference Hall*
4.15	**Close**

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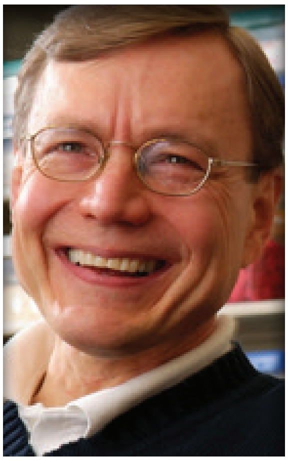
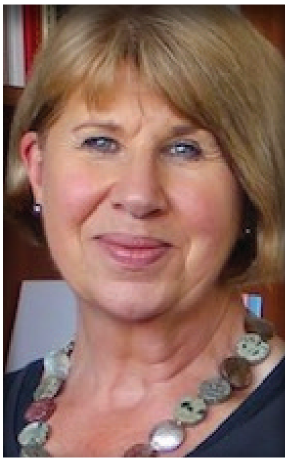
KEYNOTES



The Building Blocks of Early Mathematics

What are the building blocks of mathematics? How important are they? Doug Clements answers these questions by summarizing recent research and development work. One effective instructional approach featured in all these is basing instruction on learning trajectories. This approach will be illustrated through a set of research projects using learning trajectories successfully.

Professor Douglas Clements, University of Denver



Progression in Play and Playful Learning

Although there is much support for the value of play to children's learning and development, research shows us that teachers continue to struggle to integrate play into their practice. Often the balance between structured play and free play tips more towards the former when the priority is to achieve specific learning outcomes. Another concern is that play is left in the pre-school phase, in the transition towards more formal ways of teaching.

In this presentation, Elizabeth Wood will consider research that shows how important it is to maintain playfulness and creativity in learning and teaching. In order to do this, we need to understand progression in play - how play changes over time, and how playfulness contributes to learning dispositions that may have lifelong relevance.

Professor Elizabeth Wood, Sheffield University

MORNING WORKSHOPS

1. The Children's Measurement Project *

Prof Douglas Clements will outline his research into how young children develop the concept of measurement.

Professor Douglas Clements, University of Denver

2. Funds of knowledge in play *

In this workshop we will look at the ways in which children (4-7 years) bring together funds of knowledge in their play from many different contexts in their lives. These funds of knowledge reflect their wider social and cultural experiences, with a mix of creativity and imagination. Play reveals 'assemblages' of these funds of knowledge, including misconceptions and emerging concepts. We will consider the implications of these ideas for learning, and for how young children become mathematicians.

Professor Elizabeth Wood, Sheffield University

3. Teaching mathematical problem solving in the primary school: Changing behaviours

This workshop examines how teachers might teach through mathematical problem solving in the primary school. It will examine current issues in relation to teaching problem solving including examining how to foster interest in mathematical problem solving, creating appropriate mathematical problems and structuring effective mathematical problem solving lessons. Ideas in relation to teaching in the middle classes specifically will be explored.

Dr John O'Shea, Mary Immaculate College, Limerick and Pauric Stapleton, Carrigeen National School, Co. Kilkenny

Repeated in afternoon *

4. 10 questions you can ask instead ... Exploring teaching practices for improving the quality of discourse in mathematics classrooms

What is Maths talk? What does it look like? In this workshop we will explore research on the teacher's role, on changing the types of questions we ask and on the importance of selecting rich mathematical tasks. We'll use videos of classroom teaching to look inside three senior infants classrooms and explore their 'maths talk' as they explore and reason about algebra, data handling and shape and space. We'll work on writing open questions that will excite student curiosity, provoke critical thinking, elicit reflection and help students construct their own meaning for the mathematics they are studying.

Dr Aisling Leavy, Mary Immaculate College, Limerick and Amy Looney, Scoil Niamh Community National School, City West, D24

5. Facilitating young children's understanding of the 'equal' sign

This workshop will focus on research conducted with 1st class pupils in which Cuisenaire rods were used to challenge and develop their understanding of the equal sign. We will discuss the intricate roles that task, talk and tools play in the growth of young children's mathematical conceptions.

Dr Thérèse Dooley, St Patrick's College, Drumcondra and Aisling Kirwan, Holy Family National School, Rathcoole, Co Dublin

MORNING WORKSHOPS

6. Word-problems and the Gaelscoil child: An bhfuil fadhb ann?

This workshop examines the challenges of engaging with higher-order tasks in an immersion setting. The demands of mathematical problem-solving through Irish as a second language will be discussed in the context of a small study with children in a Gaelscoil.

Miriam Ryan, St Patrick's College, Drumcondra

7. Numeracy through play and everyday activities

This workshop will explore how numeracy concepts can be developed through play and everyday activities in the Free Preschool Year and how numeracy needs to be linked to children's home life and community environment as outlined in *Aistear*.

Patsy Stafford, Maynooth University, Froebel Dept of Primary and Early Childhood Education and Cathy Steenson, Little Treasures, North Wall Women's Centre, Lr Sheriff St, D1

8. Mathematics for children aged 3 years and under

Funded through the National Early Years Access Initiative (NEYAI), the Docklands Early Numeracy Project involved parents and early years practitioners using *Aistear* to improve children's (0-6 years) numeracy outcomes. This workshop focuses on mathematical learning activities for children aged 3 and under.

Dr Josephine Bleach, Early Learning Initiative, National College of Ireland, Michelle Moore, Parent Child Home Programme and Moira Ward, St Andrew's Resource Centre, Pearse St

AFTERNOON WORKSHOPS

1. The Children's Measurement Project *

Prof Douglas Clements will outline his research into how young children develop the concept of measurement.

Professor Douglas Clements, University of Denver

2. Funds of knowledge in play *

In this workshop we will look at the ways in which children (4-7 years) bring together funds of knowledge in their play from many different contexts in their lives. These funds of knowledge reflect their wider social and cultural experiences, with a mix of creativity and imagination. Play reveals 'assemblages' of these funds of knowledge, including misconceptions and emerging concepts. We will consider the implications of these ideas for learning, and for how young children become mathematicians.

Professor Elizabeth Wood, Sheffield University

3. Early number concepts

This workshop will look at the teaching of mathematics in the early years of primary. It will focus on both the discrete mathematics lesson and the embedding of mathematics across the curriculum using play-based methodologies to support all children's learning and development as advocated by *Aistear.*

Dr Lorraine Harbison and Audrey Halpin, Church of Ireland College of Education, Rathmines

Repeat workshop *

AFTERNOON WORKSHOPS

4. The Algebra Project

This workshop will examine the Algebra Project pedagogy and how children in infants to fifth class can focus on feature talk to mathematize an event.

Donna Owens, Máire Manning, Gael Scoil Thomais Daibhis, Mallow, Co Cork and Jerry Lynch, Rahan National School, Mallow, Co Cork

5. Supporting children at risk of experiencing difficulties in early mathematics

The session will examine aspects of early childhood development in mathematics and some of the difficulties that can arise in relation to special educational needs and the influence of social and economic factors. It will then look at some potential inclusive prevention and intervention strategies drawing on research conducted in Irish primary schools.

Dr Joseph Travers, St Patrick's College, Drumcondra and Órla McKiernan, PDST and St Mary's School, Greenhills Rd, Tallaght

6. Mathematics for children aged 3-4 years

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7. Khan Academy in the primary school

Martina Sexton of St Peter's Primary School in Bray, will talk about using Khan Academy with her second class pupils and will discuss the MATH-letes Challenge which was won by her fifth class last year.

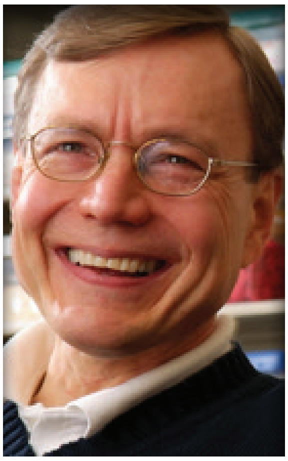
Martina Sexton, St Peter's Primary School, Bray

8. Young children mathematizing

This workshop will examine ways teachers can support young children in communicating their mathematical thinking and understanding. It will explore how strategies such as children's mark-making, drawing, use of iPad and digital photography stimulate mathematizing in the infant classroom.

Dr Elizabeth Dunphy, St Patrick's College, Drumcondra and Ross O'Corráin, Citywest Educate Together National School

KEYNOTE SPEAKERS



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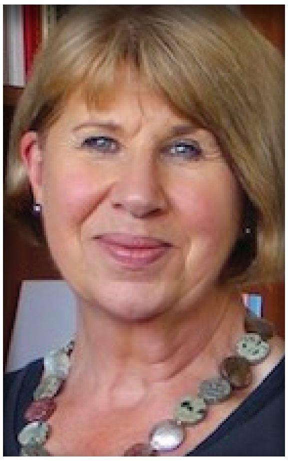

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INTRODUCTORY REMARKS

Dr. Sarah FitzPatrick, Deputy CEO

EARLY MATHEMATICS IS SURPRISINGLY IMPORTANT AND COGNITIVELY FUNDAMENTAL

Dublin Castle Conference Centre, 24th November 2014

ncca.ie/mathsconf #maths3to8

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EARLY MATHEMATICS IS SURPRISINGLY IMPORTANT AND COGNITIVELY FUNDAMENTAL

Dr. Sarah FitzPatrick, Deputy CEO

Dublin Castle Conference Centre, 24th November 2014

Dia daoibh ar maidin! Tá céad míle fáilte romhaibh ar fad go dtí Caisleán Bhaile Átha Cliath, chuig an ócáid *thar-a-bheith-speisialta seo*. You're very welcome to this special conference on developing mathematical ideas with young children.

My six-year old suggested I'd start with a story. He said, 'Mum, I think everyone likes stories... even big people. His sister who's eight said, 'I think if they're very busy adults they might need a story and if there are teachers there, someone should read to them for a change!' So here's a conundrum... Today's story is not yet written! We're here today to begin an exciting, new chapter for the Primary Mathematics Curriculum. Today's conference is about asking:

> *What do good mathematical experiences for children look like?*
> *Why are these experiences important?*
> *How can this information help shape the new*
> *Primary Mathematics Curriculum?*

Our Primary School Curriculum was developed in the 90s and published, here in Dublin Castle, 15 years ago, in 1999. It was developed in a different time, for a time yet unknown. 15 years on, our classrooms are more diverse, and teaching is more complex and demanding; we know more about children's learning and development, and appropriate early years pedagogy; significant policy developments have changed landscape of Early Childhood Education and Care and Primary Education; and we have a much greater understanding and appreciation of the importance of early years mathematics. We know that Mathematics education is not just about solving problems. Mathematics is a way of thinking, and seeing the world.

We've had some big Development Group meetings in NCCA over the years, but I think we're making history today leveraging the expertise and experience of almost 300, involved in Early Years Education and Mathematics Education towards curriculum development.

Today is historic in bringing together practitioners/teachers and researchers, who traditionally, have had their own, culture, heritage and language. Our goal today, is to connect research and practice towards development of a new Primary Mathematics Curriculum.

There is no 'holy grail' no 'silver bullet', in education. But I think there are some vital ideas - clear signposts for curriculum improvements at primary. These vital ideas are informed by findings from curriculum reviews and evaluations; national and international assessments;

work with schools and settings; and the Mathematics research reports being launched here today. I'd like to share three vital ideas with you which I believe are significant for the next phase of curriculum revision at Primary. These vital ideas are: *Passion, Practice* and *Pathways.*

Passion

Children are already mathematicians before pre-school or primary. Both young children and mathematicians ask and think about deep questions, use maths to solve everyday problems, and play with maths. Young children already see the world through maths glasses. They are natural risk-takers in their thinking (and actions!) Maths is part of the DNA of children's conversations, play, daily routines and interactions. They're using maths when they:

- Set the table for the teddy bear's picnic and share out the treats
- Build with blocks; play floating and sinking games.
- Count and use numbers to label things
- Describe things as long, short, heavy, light, and spot and extend patterns.

EARLY MATHEMATICS IS SURPRISINGLY IMPORTANT AND COGNITIVELY FUNDAMENTAL

We know that children come to school with their own ideas and concepts about the world, their 'funds of knowledge', which they share in stories about the biggest dinosaur, the best adventure, why their house has a number, and what we do with mobile phones and tablets. Children's tremendous curiosity and capacity for thinking are evident in questions about:

- What things cost and how much things weigh
- How far it is to the cousin's house
- How much I've grown
- If the shape we sit in, affects our thinking
- Do our thoughts go round and round when we're in circles?
- If we sit at a triangle table is it easier to come to the point?
    - And at this time of year... how many sleeps till Santa comes?!
    - Count and use numbers to label things
- Describe things as long, short, heavy, light, and spot and extend patterns.

These examples of mathematical experiences in young children's lives remind us that for young children, maths is both content and process. The research reports to be launched today tell us that concepts and processes must be combined in the new Primary Mathematics Curriculum. The eight processes to flow through teaching and learning are:

- Connecting
- Communicating
- Reasoning
- Argumentation
- Justifying
- Representing
- Problem-solving
- Generalising.

So the new Primary Mathematics Curriculum will use rich mathematical tasks to combine learning processes and concepts. It will help teachers to connect with parents, in order to build-on children's passion for learning, and their capacity to take risks in their thinking. In short, it will help children to develop productive dispositions for a lifetime of maths learning.

Practice

We turn now to teachers' dispositions. In a review of the maths curriculum (NCCA, 2005), we reported that the teachers' attitude towards maths directly influenced the extent to which children liked or disliked maths:

Child 1 My favourite part of maths is all of it!
Child 2 Yeah, we all like maths.
NCCA So there's no part of maths you dislike?
Child 3 But our teacher loves it, so we love it too!

I think of positive disposition and passion as *fuel* to drive the learning journey. We've all had an experience of doing something that didn't light our fire and on the other hand, we've experienced being thoroughly engaged, being 'in-flow', when we've been living 'in the moment' in the fullest, possible sense (Gaffney, 2011, p.277).

Revisions to the primary curriculum, must also fire-up teachers' passions. Building-on our work with schools and Early Years settings, we know that high quality support material which shows what good teaching/learning *looks and sounds* like across a range of settings is one key to ongoing learning and improvement for teachers as well as children. You'll see some videos shortly, introduced by Arlene, and indeed through the day, which show rather than talk about what children do, say and make when learning maths through their interactions with others and the environment, through play, talk and rich tasks. The '99 curriculum pre-dated online toolkits, which now have the power to demystify pedagogy and to make explicit in practice, some tacit messages about theories of teaching and learning for teachers, also relevant to parents and learners themselves.

The new Primary Mathematics Curriculum will build-on the descriptions of the teacher's role in *Aistear, the Early Childhood Curriculum Framework* (NCCA, 2009, Guidelines, p.28-30), showing how it changes, depending on the extent to which the child or the teacher is leading learning.

The new Primary Mathematics Curriculum will also demystify standard by including examples of children's work which show their maths learning along with teachers' comments on achievement.

Pathways toward learning destinations

For both teachers and learners, being clear on where the learning is going, on the learning outcomes, is a first step to being able to talk about progress and achievement relative to our expectations for learners. When teachers identify and share their learning intentions with children, they empower children to be leaders of their learning. But knowing the learning destination—the intended outcome—isn't enough. The research reports to be launched today describe how learning paths or pathways informed by research and development have replaced the idea of stages of development in children's maths learning (generally associated with the work of Piaget). Learning pathways show important milestones in the learner's trajectory or journey toward the learning destination. Knowing these learning milestones helps practitioners/teachers to understand the sequence, in a general sense, of children's learning and development across maths domains and to support *all* learners. How tricky it would have been to get to this conference centre in Dublin Castle today, if we didn't have a map or any set of directions! For practitioners/teachers, access to Information about learning maps is vital for the same reasons. This is one significant change in the new Primary Mathematics Curriculum.

Another change? The new Primary Mathematics Curriculum will support continuity and cohesion for learners in their learning journey before and after primary school by connecting the curriculum with learning goals and outcomes at pre-school and post-primary. We have a new curriculum specification which uses a similar structure across primary and post-primary giving us a common language for a shared agenda. In the past, curriculum policy has often been developed in a sectoral approach.

The joined-up approach with one curriculum specification for primary and post-primary recognises that learners themselves do not live in sectors. It also acknowledges that vital ideas in education like passion/engagement; supporting teachers' practice; and pathways to progression are valued across sectors.

So, to conclude, we know that children need a curriculum which engages, empassions, and challenges them to meet high expectations. Teachers need a less-crowded curriculum which provides clear roadmaps and destinations for learning and which supports them to work with parents to connect children's learning in school with their lives out-side school, and to plan and provide high-quality learning opportunities for all children to progress in their mathematics learning. We can expect a curriculum that engages children's *passion* for learning, supports teachers' *practice* and provides clear *pathways* and destinations for learning.

And so today, after the launch of the two commissioned research papers on mathematics in the early years (age 3-8) by Jan O' Sullivan, T.D., Minister for Education and Skills, we'll begin developing the Primary Maths Curriculum. There are three big milestones in the work ahead:

- Engagement and consultation on the Primary Mathematics Curriculum for the four junior classes will begin next autumn.
- The Primary Maths Curriculum (for first four primary years (junior infants to 2nd class) will be published in September 2016, and
- The Primary Maths Curriculum for the senior primary classes (3rd to 6th class) will be published approximately two years later in 2018.

Before I had-over to Arlene now to tell us about what's in store today, it's my pleasure to extend a very warm welcome to everyone here today, to:

- Early years practitioners and teachers; managers and principals
- Third-level educators, and researchers and
- Those involved in teacher education, training and professional development and policy.

I'd like to say a very special welcome and thanks to:

- All our workshop presenters today
- Authors of the commissioned research reports on mathematics for children aged 3-8 to be launched later today: Drs. Elizabeth Dunphy and Therese Dooley, St. Patrick's College, Drumcondra and Dr. Gerry Shiel, Educational Research Centre (and their co-authors), and
- Our keynote speakers Profs Doug Clements (University of Denver) and Elizabeth Wood (Sheffield University).

I wish you all an enjoyable and inspiring day and I want to thank you for contributing to the next chapter on the Primary Mathematics Curriculum with us beginning today. Tá súil agam go mbainfidh sibh taitneamh agus tairbhe as an lá! Go raibh míle maith agaibh.

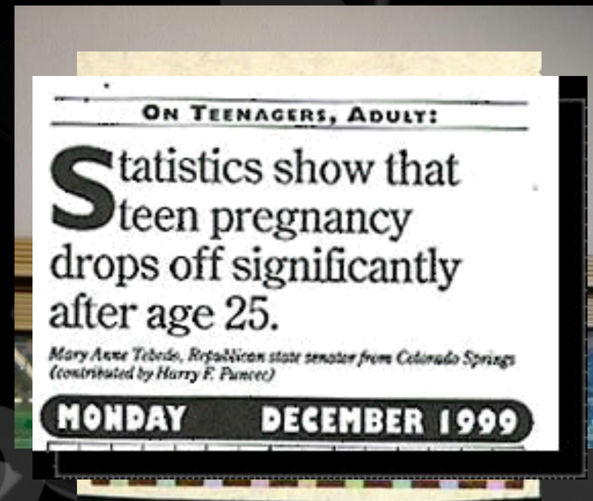
Early Math: Surprisingly Important

Douglas H. Clements
University of Denver

Building Blocks

1

We Need Better Math



ON TEENAGERS, ADULT:

Statistics show that teen pregnancy drops off significantly after age 25.

Mary Anne Tebedo, Republican state senator from Colorado Springs (contributed by Harry F. Puncec)

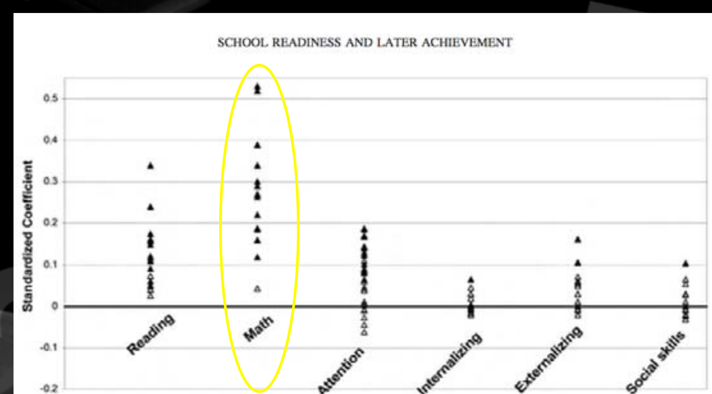
MONDAY DECEMBER 1999

2

3

Surprise #1: Math's Predictive Power

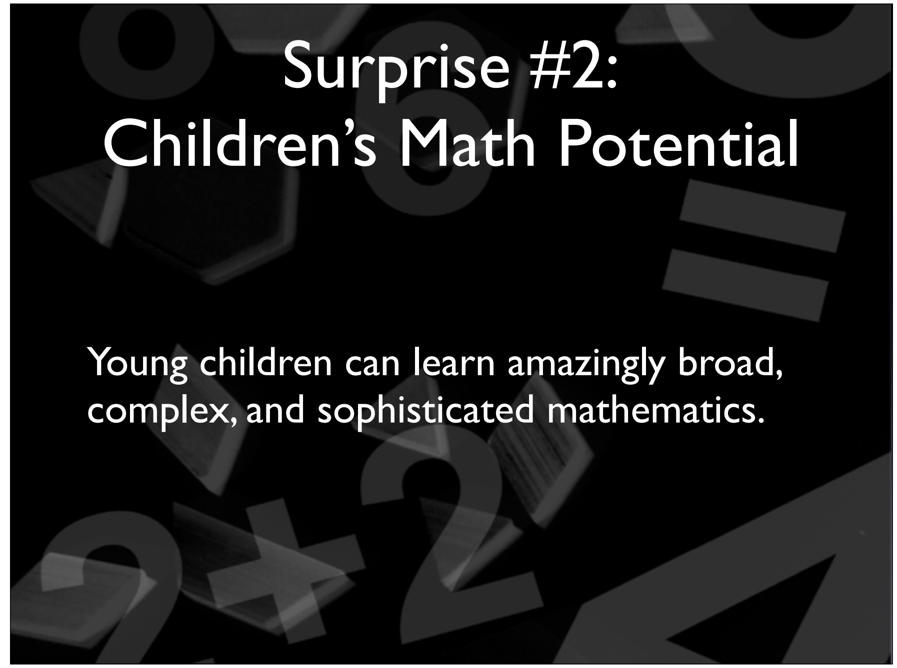
Large-scale research, predicting school success
(Duncan et al., 2007)



4


Foundations for Success
"Most children acquire considerable knowledge of… mathematics before … kindergarten.
…related to their mathematics learning for years thereafter - even high school."
MATHEMATICS in Early
Foundations for
Success
The National Mathematics Advisory Panel
Final Report • 2008

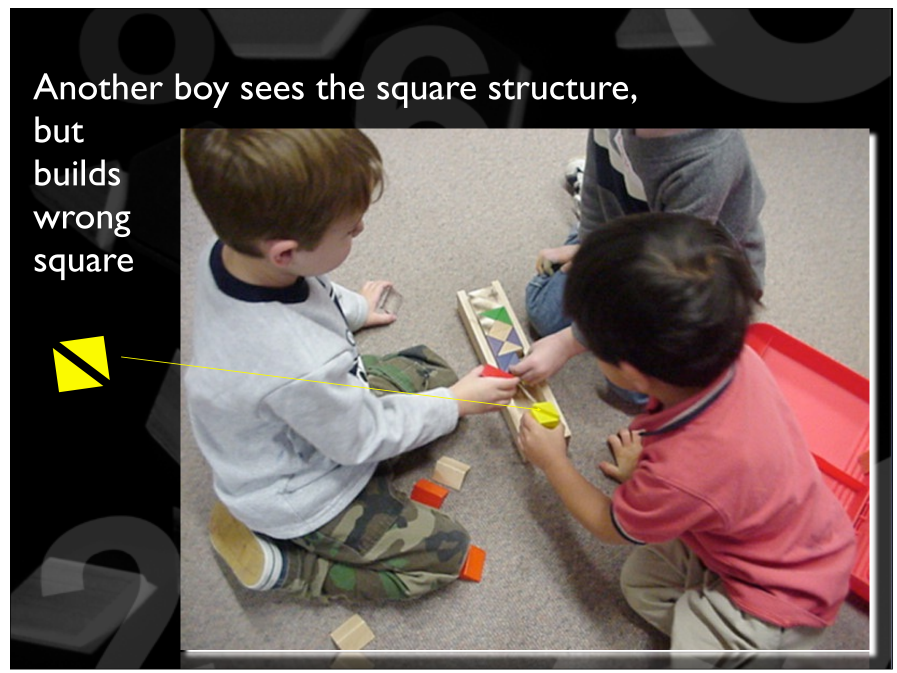
5


Surprise #2:
Children's Math Potential
Young children can learn amazingly broad, complex, and sophisticated mathematics.

6


Preschool Potential
Cory makes a new shape:
A unit of units
Cory is putting 4 triangles together to make squares

7


Another boy sees the square structure, but builds wrong square

8

Finishing, Cory shows adult, who asks:
"How many triangles did you use?"

Cory counts: "24"

"24 what?"

"Triangles."

"How many squares do you have?"

Puts 4 fingers on triangles in each new unit and counts each square: "6!"

9

Third Grade Subtraction

- Reintroduction:
  64
  -28
- "Now, I can't take 8 from 4, so…"

  -Kye, a 3rd grade boy, interrupted:
- Yes, you can! 8 from 4 is negative four
  64
  -28
  -4

10

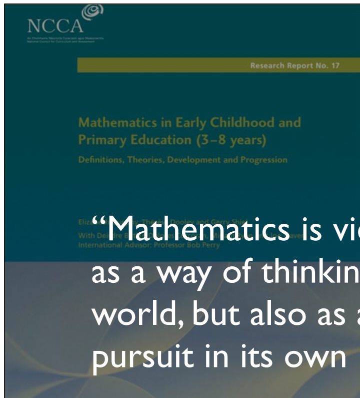
- "…and 20 from 60 is forty:
  64
  -28
  -4
  40
- …and negative four and 40 is 36"
  64
  -28
  -4
  40
  36

11

"In the curriculum, a view of all children as having the capacity to engage with deep and challenging mathematical ideas and processes from birth should be presented."

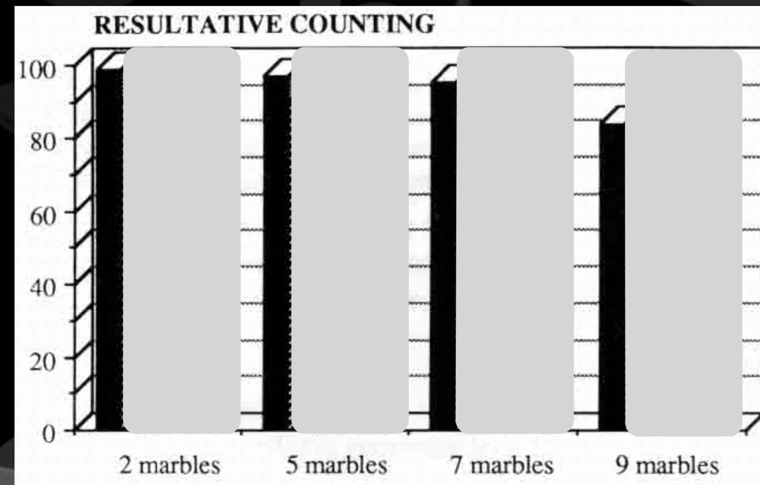
— Primary Maths Reports…

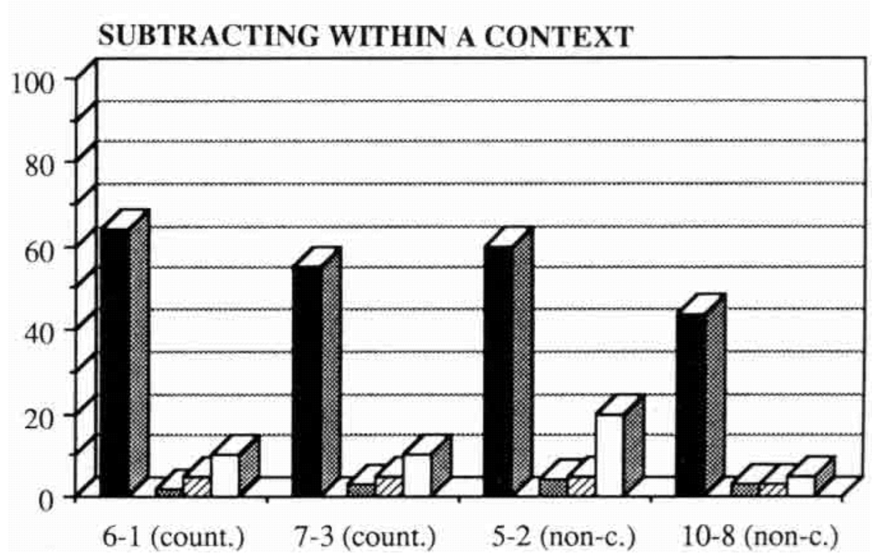
12



"Mathematics is viewed not only as useful and as a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004).

All children are viewed as having an ability to solve mathematical problems, make sense of the world using mathematics, and communicate their mathematical thinking."





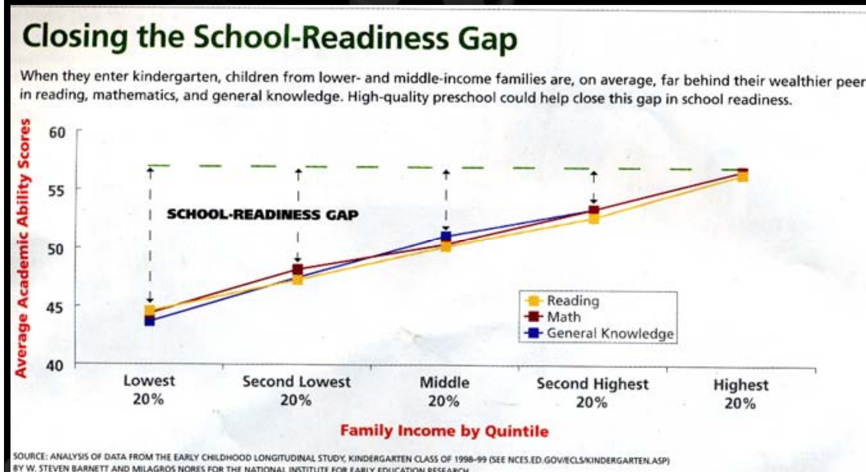
Surprise #3: Surprise to Educators

- What young children can learn is a surprise to most early/primary educators.
- Therefore, they do not challenge children or use formative assessment effectively (especially "ends").

Surprise #4:
Most Children Need a Math Intervention

17

Not Just the Poor

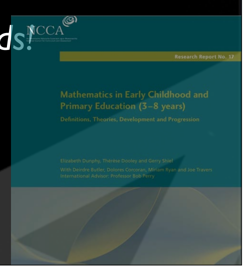


18

Equity

- Especially low-income children who hear about 1,500 number words a year
- Compare to middle-income: 93,000
- Middle hear *60 times as many words!*

'mathematics for all' —



19

Surprise #2 Revisited: Children's Math Potential

Children *invent* mathematics…

20


How much is 9 take away 4?

21


Why Not Just the Algorithm?

22

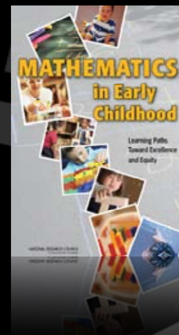
23


1000 - 359
999 - 359

24

Surprise #5: We Know a *Lot*

- About how children think about and learn math
- *Learning trajectories*



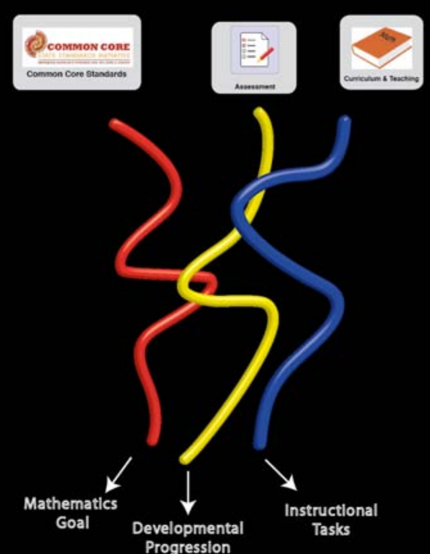
COMMON CORE
STATE STANDARDS INITIATIVE
PREPARING AMERICA'S STUDENTS FOR COLLEGE & CAREER

25

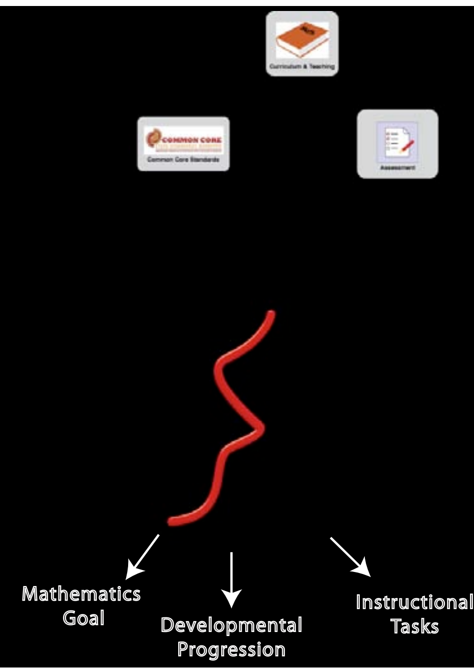
Learning Trajectories: 3 Parts

1. Goal
2. Developmental Progression
3. Instructional Activities

26



27

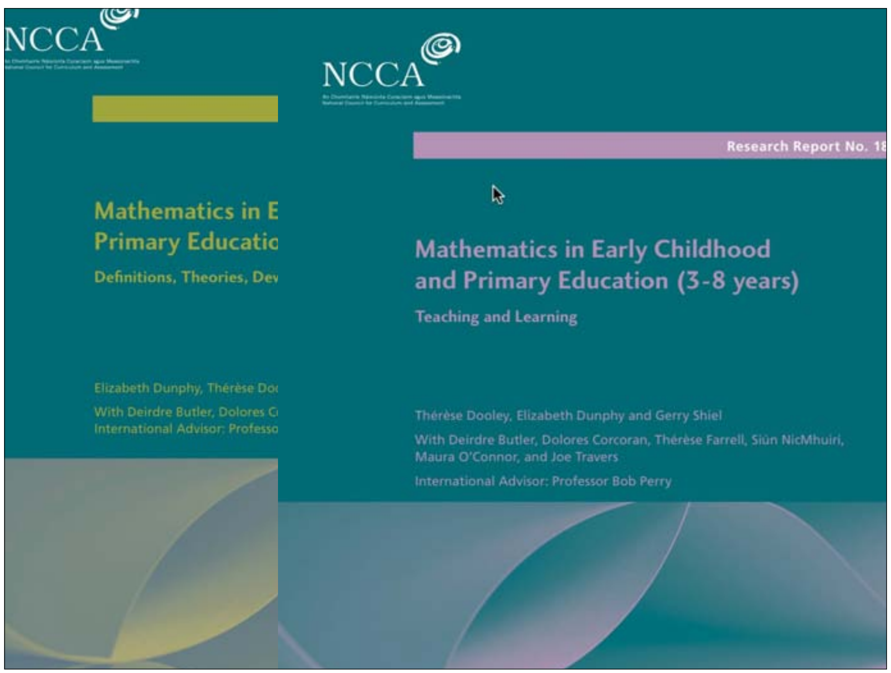


Scientific Approach to Learning Trajectories weaves the 3 parts together

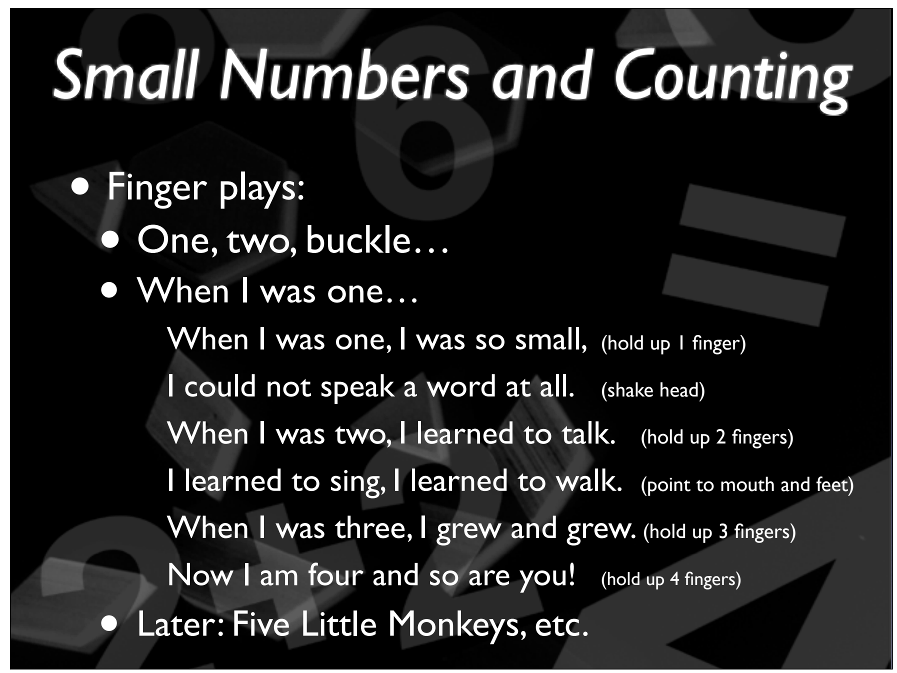
28

29

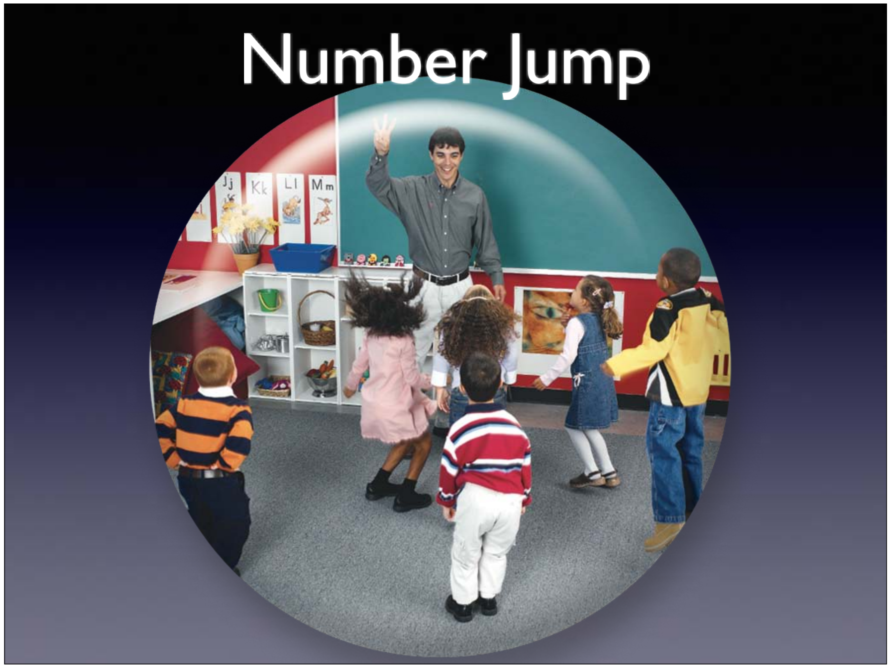
30


NCCA
Mathematics in E
Primary Educatic
Definitions, Theories, Dev
NCCA
Research Report No. 18
Mathematics in Early Childhood
and Primary Education (3-8 years)
Teaching and Learning
Thérèse Dooley, Elizabeth Dunphy and Gerry Shiel
With Deirdre Butler, Dolores Corcoran, Thérèse Farrell, Siún NicMhuirí,
Maura O'Connor, and Joe Travers
International Advisor: Professor Bob Perry

31

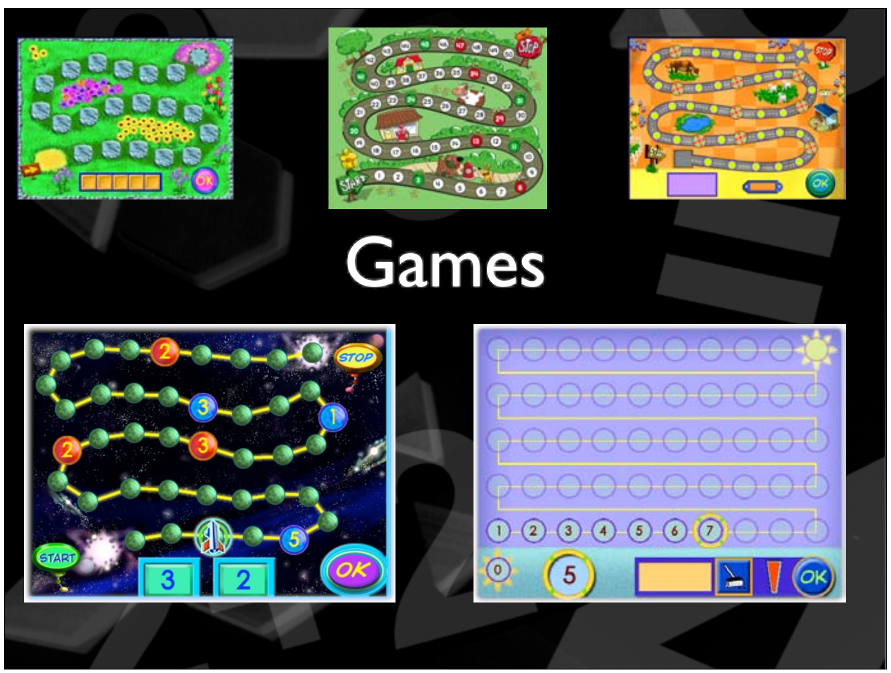

Small Numbers and Counting
Finger plays:
One, two, buckle…
When I was one…
When I was one, I was so small, (hold up 1 finger)
I could not speak a word at all. (shake head)
When I was two, I learned to talk. (hold up 2 fingers)
I learned to sing, I learned to walk. (point to mouth and feet)
When I was three, I grew and grew. (hold up 3 fingers)
Now I am four and so are you! (hold up 4 fingers)
Later: Five Little Monkeys, etc.

32



33

34



35

36

Road Race:
Counting in Two "Worlds"

- Count the dots and move that number of jumps
- *Connecting* different concepts of number

37

Road Race Shape Counting - Another Variation

- Count the sides of a shape and move that number of jumps
- Connecting new concepts of number

38

Space Race Number Choice

- Choose the "better" of two numbers
- Comparing but also reasoning: Which is better in this case?

39

Arithmetic Sequence

Encourage counting on from numeral
Add numerals

Addition "choice" game

Two-digit addition

40


Surprise #5: We Know a *Lot*
About how to *scale up* teaching and learning with learning trajectories
Foundations for Success
The National Mathematics Advisory Panel
MATHEMATICS in Early Childhood
TRIAD
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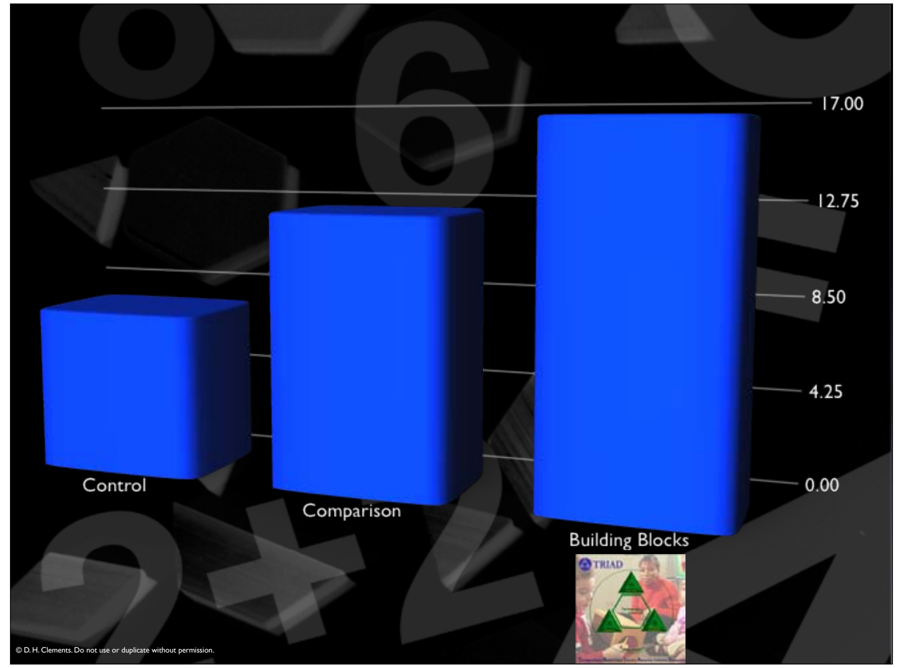
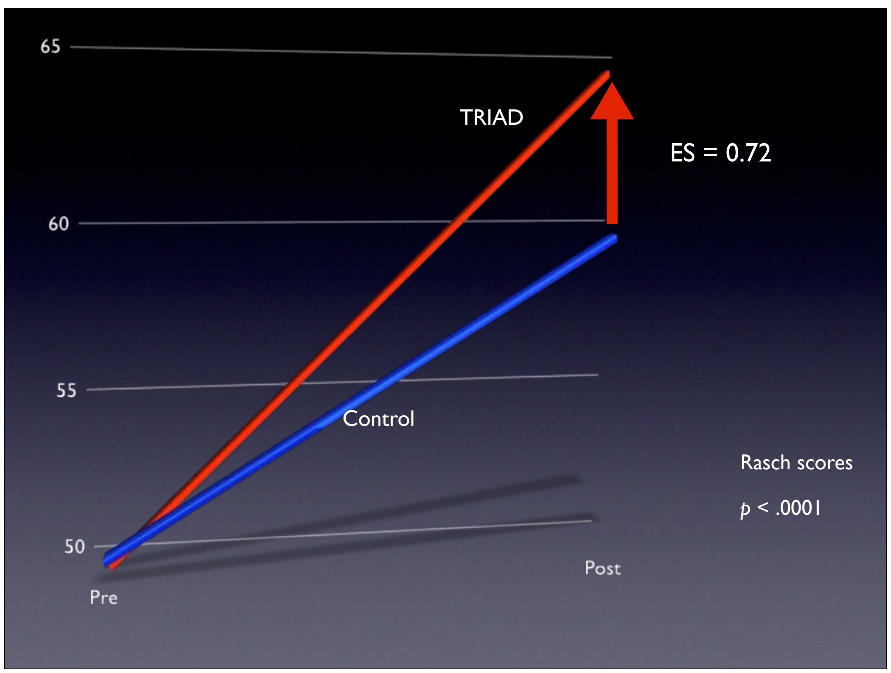
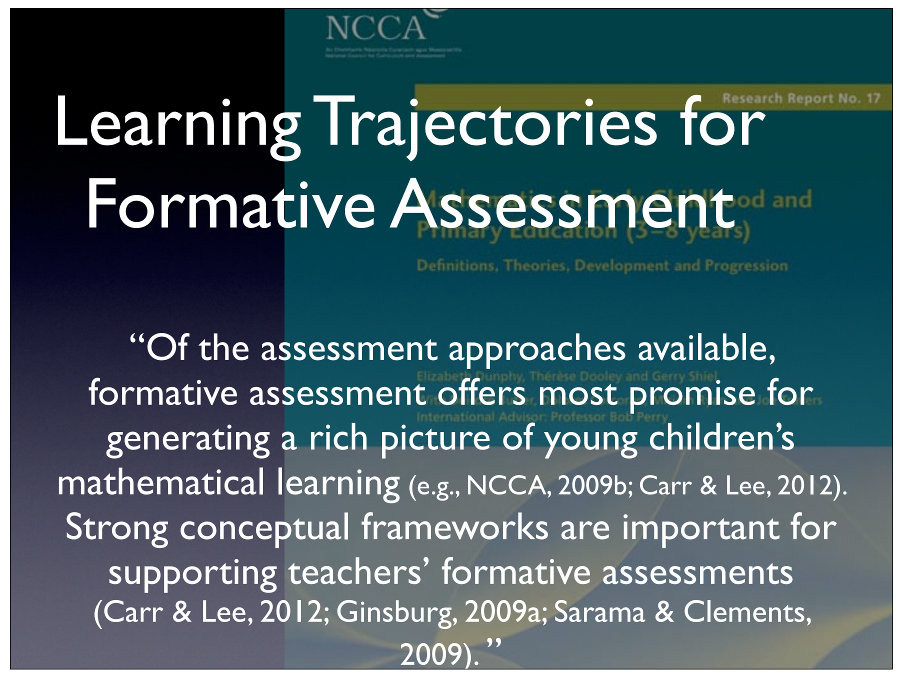
41


17.00
12.75
8.50
4.25
0.00
Control
Comparison
Building Blocks
TRIAD
© D.H. Clements. Do not use or duplicate without permission.

42


65
60
55
50
TRIAD
ES = 0.72
Control
Rasch scores
$p < .0001$
Pre
Post

43


NCCA
Research Report No. 17
Learning Trajectories for Formative Assessment
Definitions, Theories, Development and Progression
"Of the assessment approaches available, formative assessment offers most promise for generating a rich picture of young children's mathematical learning (e.g., NCCA, 2009b; Carr & Lee, 2012). Strong conceptual frameworks are important for supporting teachers' formative assessments (Carr & Lee, 2012; Ginsburg, 2009a; Sarama & Clements, 2009)."

44

Surprise #5a: Math + Play

Building mathematics knowledge does not require sacrificing play.

45

Important: Only an *intuitive foundation!*

Mathematical Activity in PreK Play

Average percentage of minutes in which mathematical activity occurred:

42%

46

Math, Literacy, and Play

- Curricula focus lead to stronger emphasis in subject-matter
- Children in content-focused classrooms *more* likely to engage at high-quality level during free play
- Those focusing on *both math and literacy* more engaged at high level than neither or only one!

47

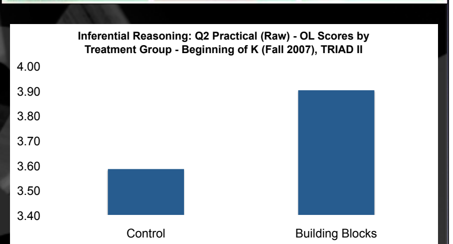
Play with Ideas

- Regular play with blocks, puzzles, socio-dramatic play (with self-regulation), *and*
- *Enhancement of math* in that play, *and*
- Intentional, planned, math (LTs)...*and*
- *Play with mathematics*

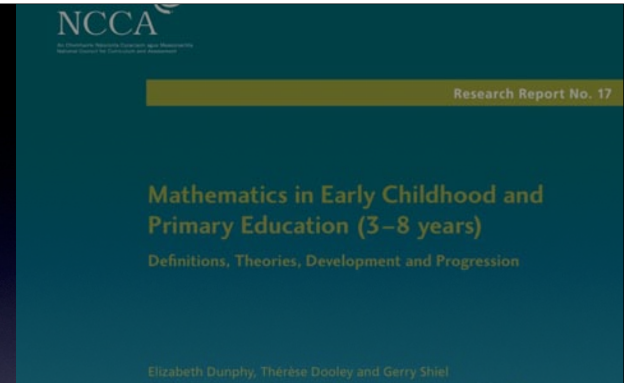
48

Surprise #5b: Language and Literacy Do Not Suffer

- *No difference on letter naming or 3 expressive language measures.*
- *Sig. <u>higher</u>* for TRIAD on:
    - Information
    - Complexity
    - Independence
    - Inferential Questions



49

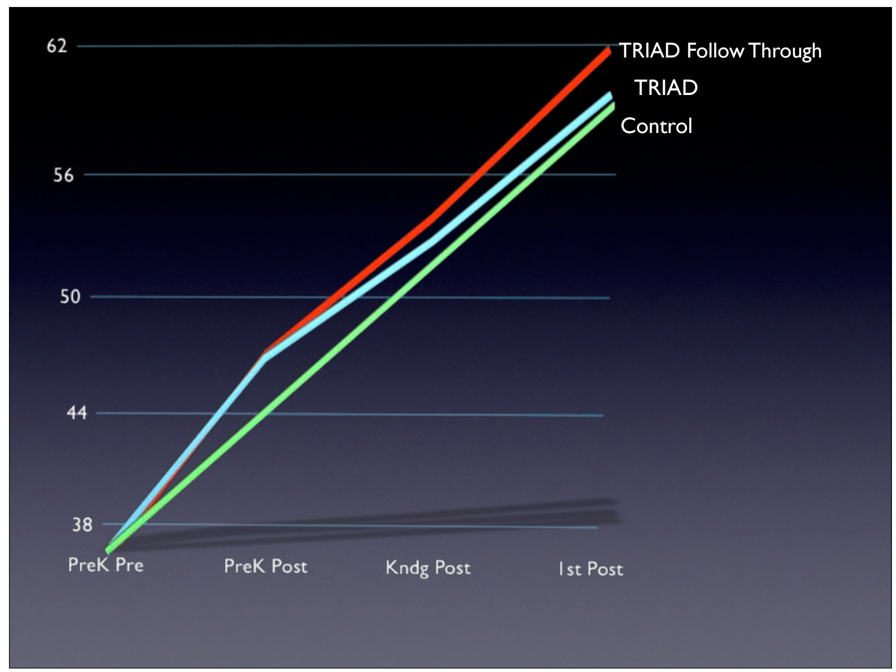


"In addition to introducing young children to mathematical vocabulary, it is important to engage them in 'math talk' – conversations about their mathematical thinking and reasoning ."

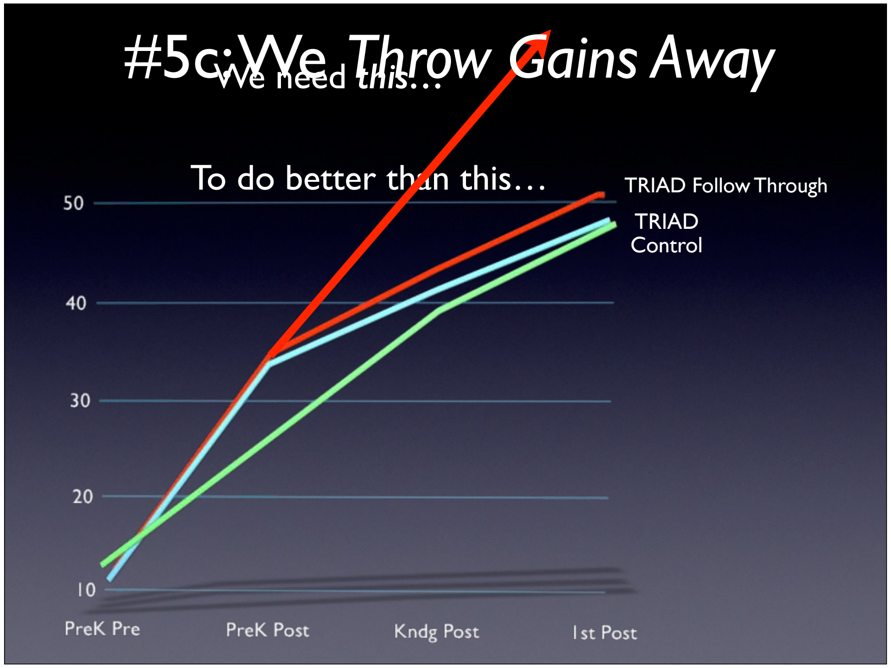
50

Building Blocks of Math

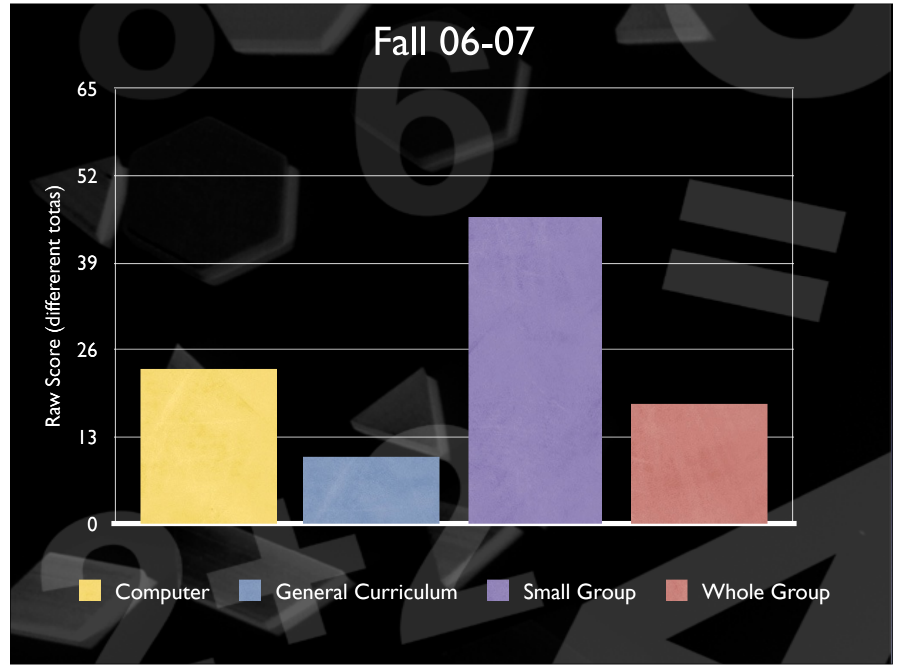
51



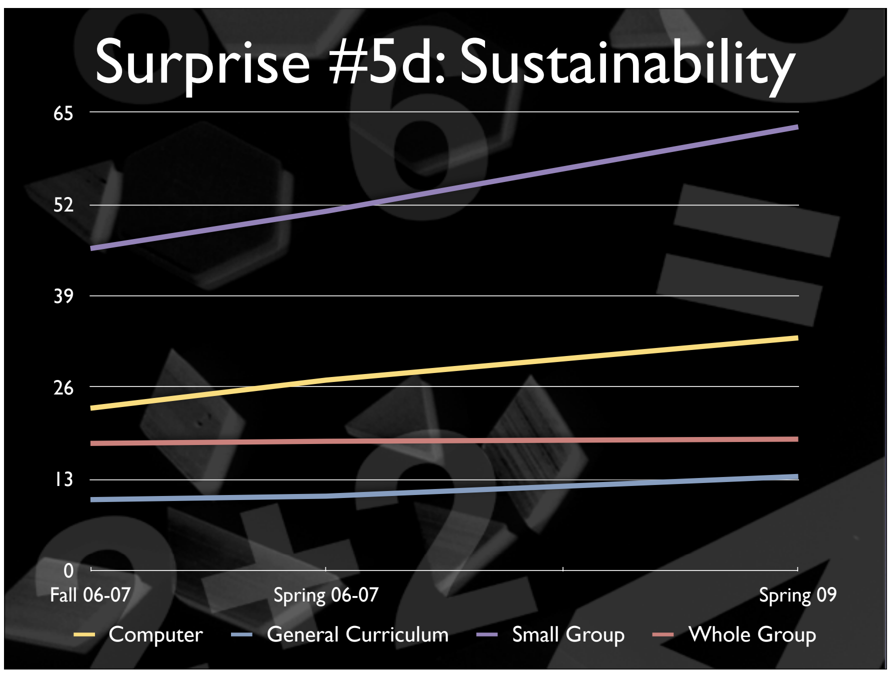
52


#5c: We Throw Gains Away
We need this…
To do better than this…
TRIAD Follow Through
TRIAD
Control
50
40
30
20
10
PreK Pre
PreK Post
Kndg Post
1st Post

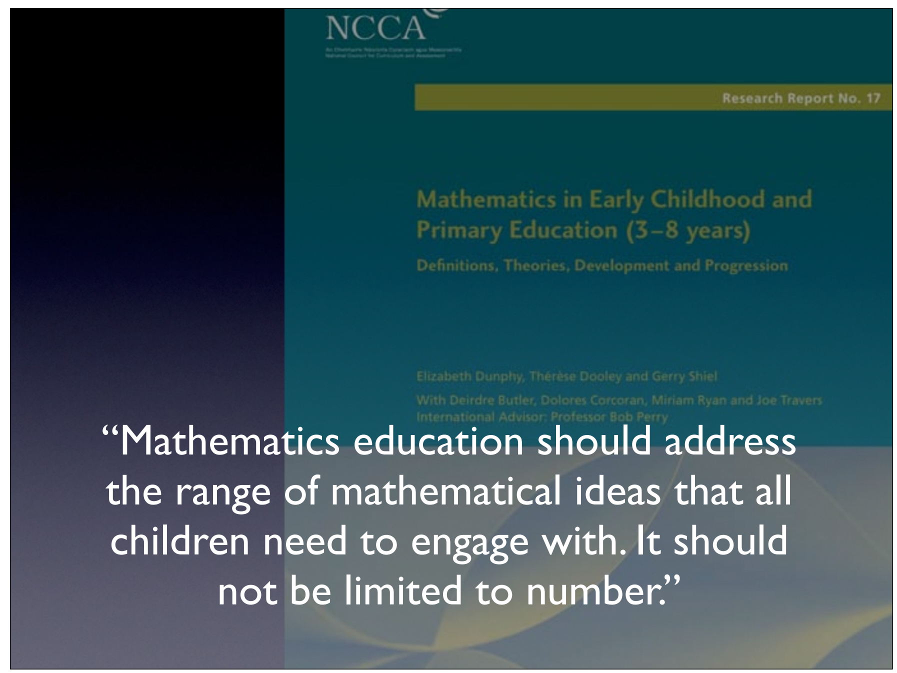
53


Fall 06-07
Raw Score (differerent totas)
65
52
39
26
13
0
Computer
General Curriculum
Small Group
Whole Group

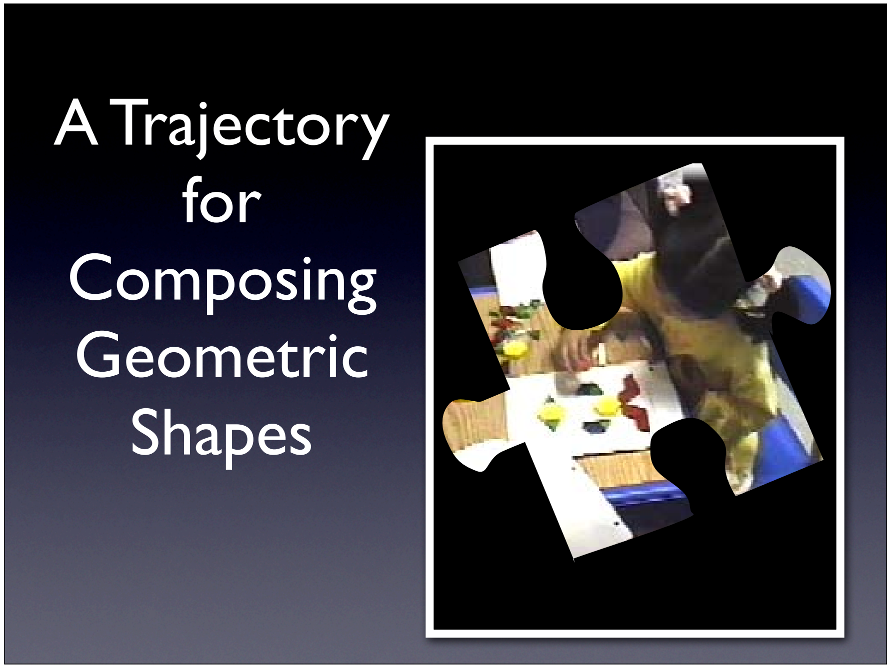
54


Surprise #5d: Sustainability
65
52
39
26
13
0
Fall 06-07
Spring 06-07
Spring 09
Computer
General Curriculum
Small Group
Whole Group

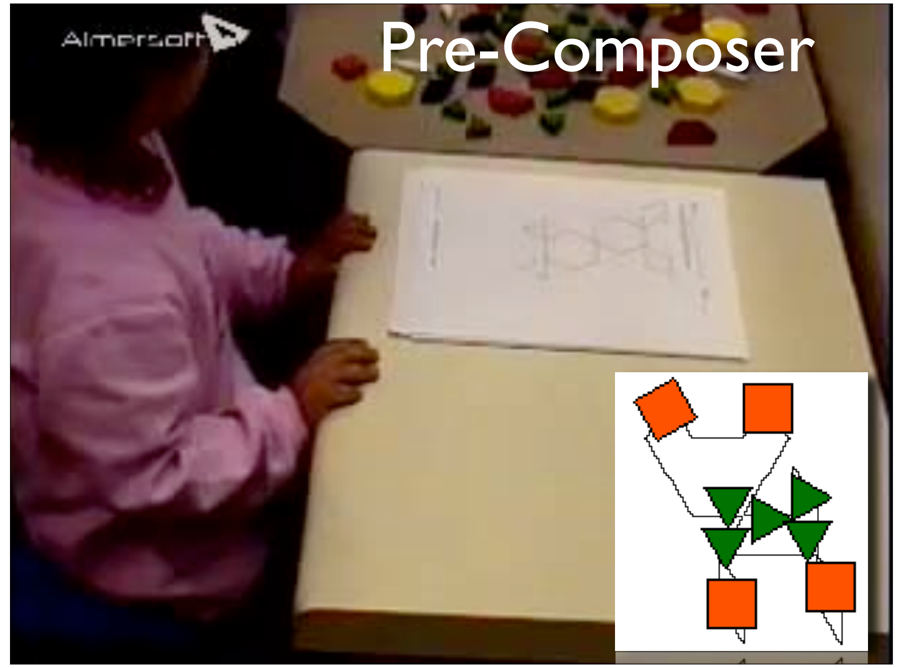
55


NCCA
Research Report No. 17
Mathematics in Early Childhood and Primary Education (3–8 years)
Definitions, Theories, Development and Progression
Elizabeth Dunphy, Thérèse Dooley and Gerry Shiel
With Deirdre Butler, Dolores Corcoran, Miriam Ryan and Joe Travers
International Advisor: Professor Bob Perry
“Mathematics education should address the range of mathematical ideas that all children need to engage with. It should not be limited to number.”

56


A Trajectory
for
Composing
Geometric
Shapes

57


Pre-Composer

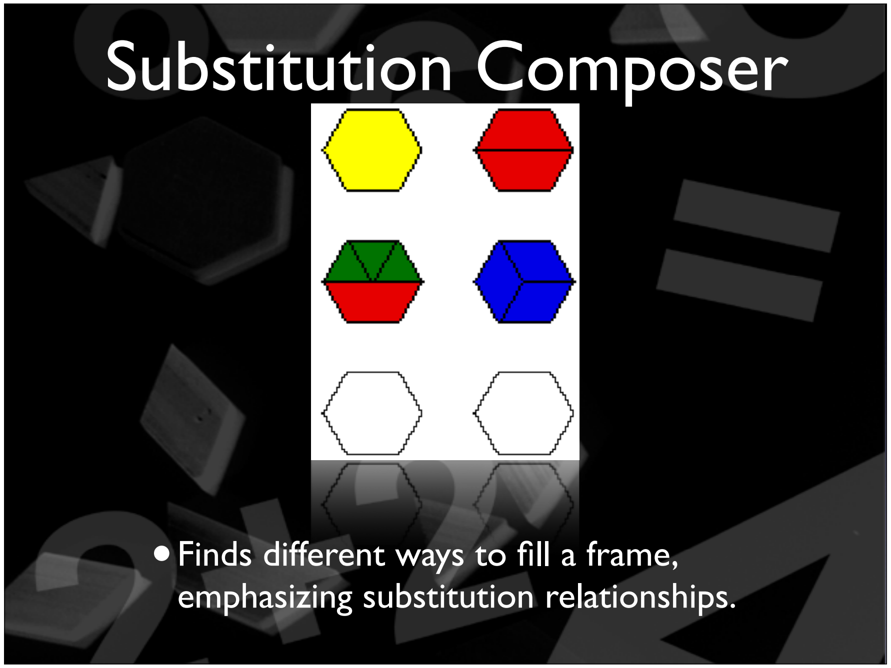
58


Picture Maker

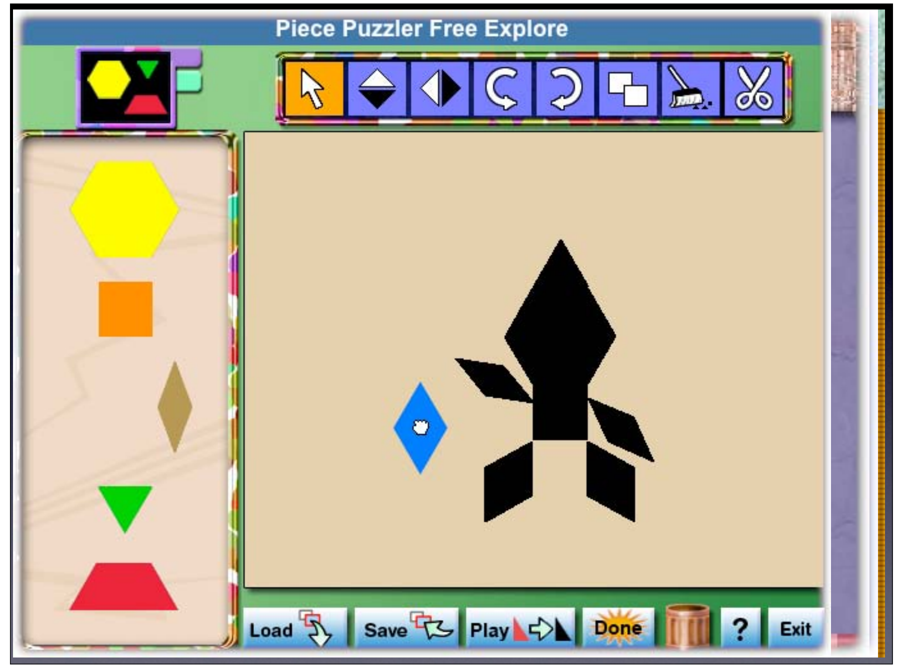
59


Shape Composer
What's important?
—Think, pair, share

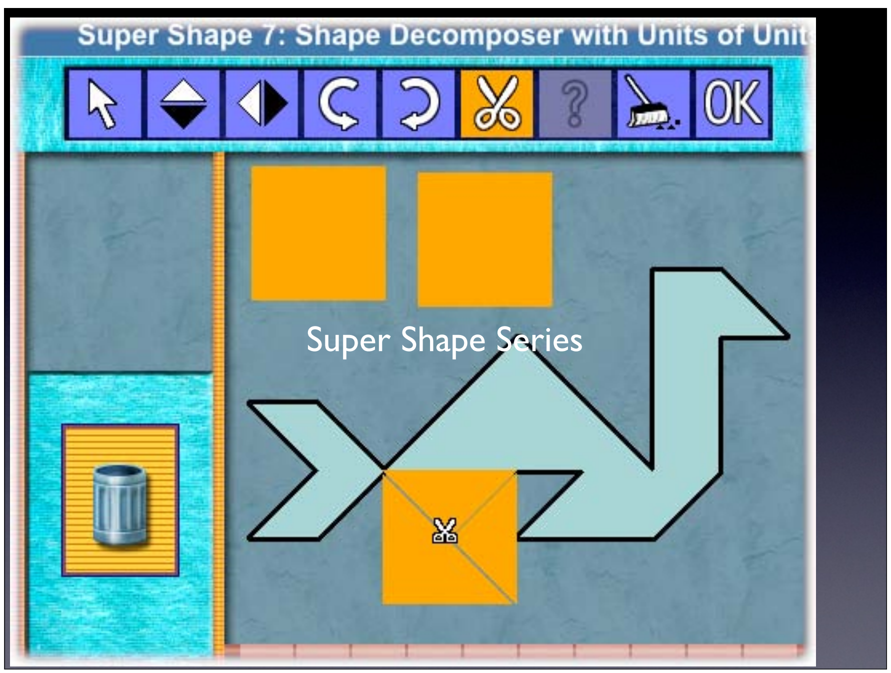
60


Substitution Composer
• Finds different ways to fill a frame, emphasizing substitution relationships.

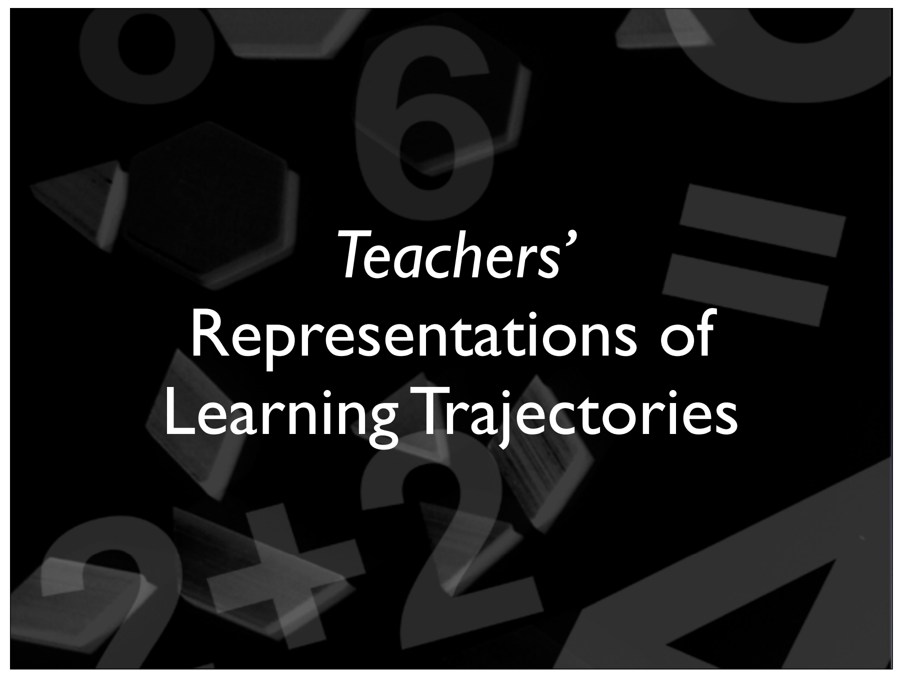
61


Piece Puzzler Free Explore
Load
Save
Play
Done
?
Exit

62


Super Shape 7: Shape Decomposer with Units of Unit
OK
Super Shape Series

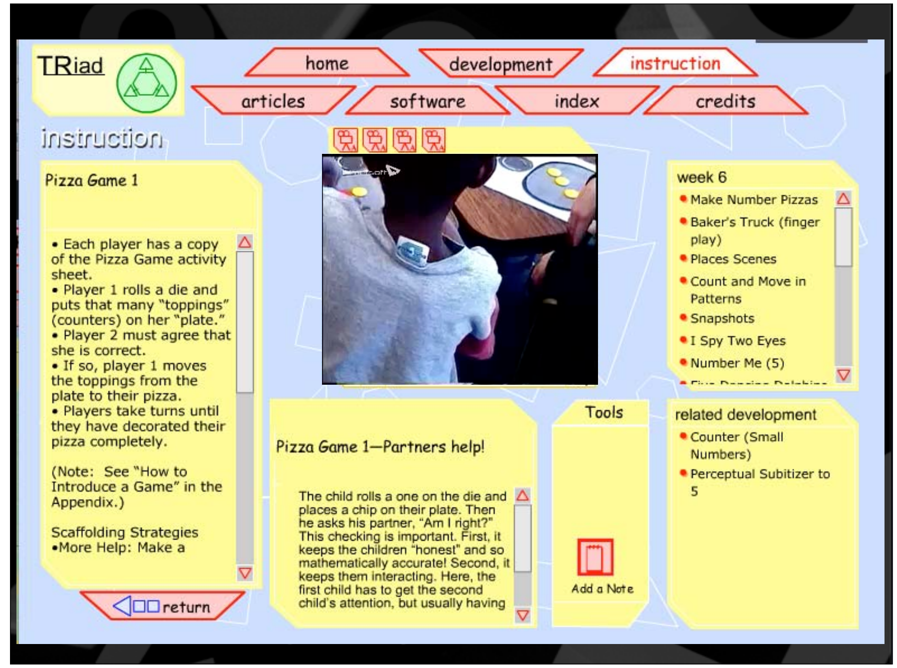
63


Teachers'
Representations of
Learning Trajectories

64


TRiad
home
development
instruction
articles
software
index
credits
week 6
instruction
m t w th f
Whole Group
Make Number Pizzas
Baker's Truck (finger play)
Places Scenes
Count and Move in Patterns
Snapshots
I Spy Two Eyes
Number Me (5)
Five Dancing Dolphins
Small Group
Pizza Game 1
Make Number Pizzas
Computer Center
Pizza Pizzazz 2 (1-5)
Pizza Pizzazz Free Explore
Road Race Counting Game
Hands-on Math Center
1st Denotes first occurrence
Pizza Game 1
• Each player has a copy of the Pizza Game activity sheet.
• Player 1 rolls a die and puts that many "toppings" (counters) on her "plate."
• Player 2 must agree that she is correct.
• If so, player 1 moves the toppings from the plate to their pizza.
• Players take turns until they have decorated their pizza completely.
(Note: See "How to Introduce a Game" in the Appendix.)
Scaffolding Strategies
•More Help: Make a
More Info

65


TRiad
home
development
instruction
articles
software
index
credits
instruction
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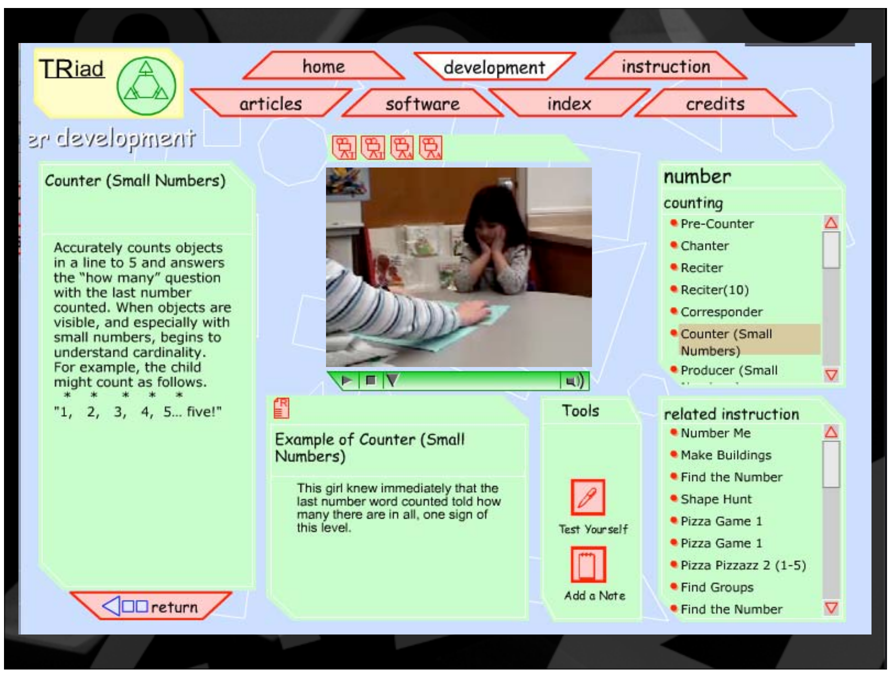
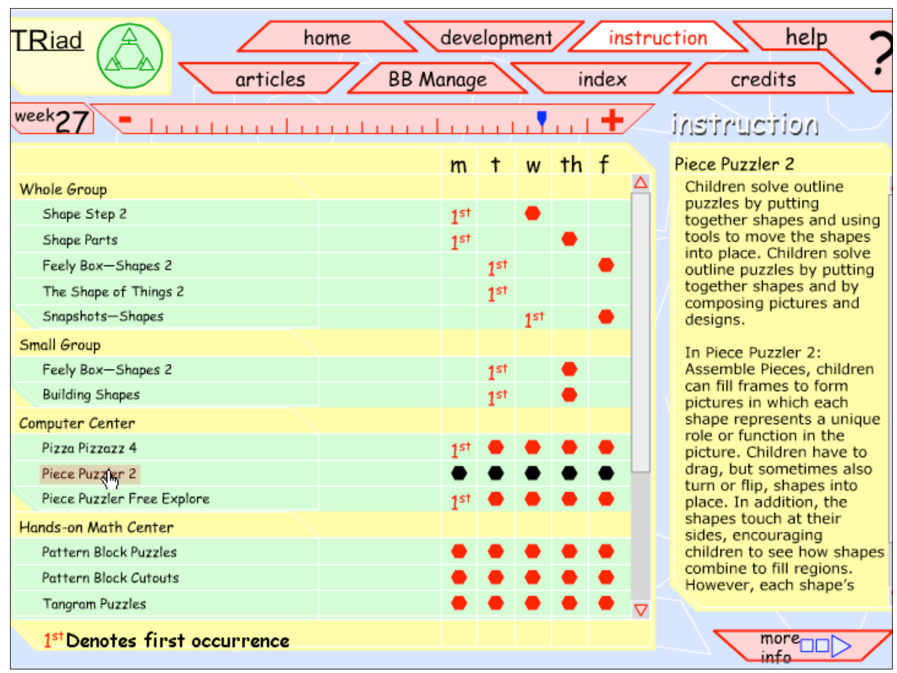
• Player 1 rolls a die and puts that many "toppings" (counters) on her "plate."
• Player 2 must agree that she is correct.
• If so, player 1 moves the toppings from the plate to their pizza.
• Players take turns until they have decorated their pizza completely.
(Note: See "How to Introduce a Game" in the Appendix.)
Scaffolding Strategies
•More Help: Make a
return
week 6
Make Number Pizzas
Baker's Truck (finger play)
Places Scenes
Count and Move in Patterns
Snapshots
I Spy Two Eyes
Number Me (5)
Pizza Game 1—Partners help!
The child rolls a one on the die and places a chip on their plate. Then he asks his partner, "Am I right?" This checking is important. First, it keeps the children "honest" and so mathematically accurate! Second, it keeps them interacting. Here, the first child has to get the second child's attention, but usually having
Tools
Add a Note
related development
Counter (Small Numbers)
Perceptual Subitizer to 5

66


TRiad
home
development
instruction
articles
software
index
credits
Counter (Small Numbers)
Accurately counts objects in a line to 5 and answers the "how many" question with the last number counted. When objects are visible, and especially with small numbers, begins to understand cardinality. For example, the child might count as follows.
"1, 2, 3, 4, 5... five!"
return
Example of Counter (Small Numbers)
This girl knew immediately that the last number word counted told how many there are in all, one sign of this level.
Tools
Test Yourself
Add a Note
number
counting
Pre-Counter
Chanter
Reciter
Reciter(10)
Corresponder
Counter (Small Numbers)
Producer (Small
related instruction
Number Me
Make Buildings
Find the Number
Shape Hunt
Pizza Game 1
Pizza Game 1
Pizza Pizzazz 2 (1-5)
Find Groups
Find the Number

67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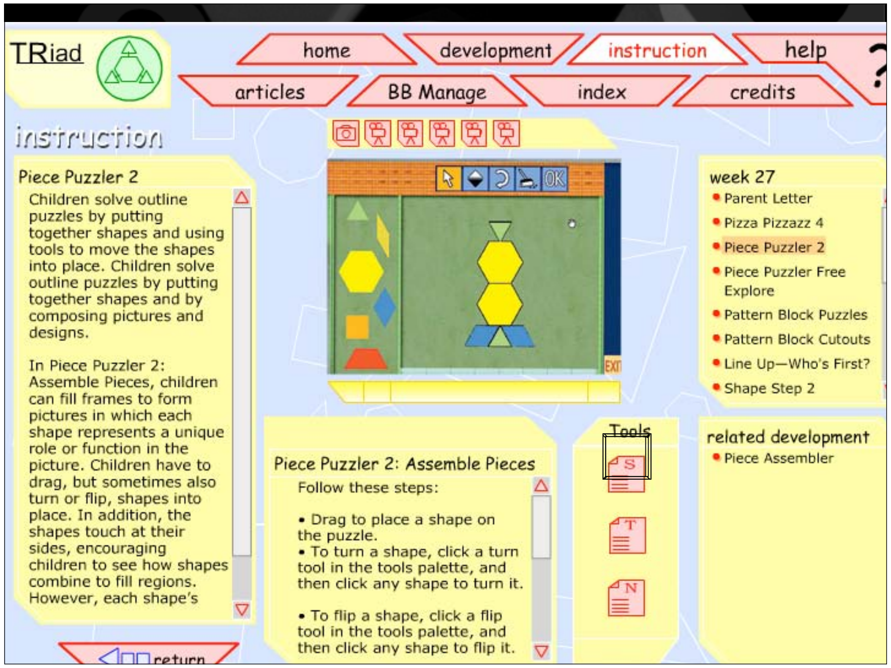
help
articles
BB Manage
index
credits
week 27
instruction
m t w th f
Whole Group
Shape Step 2
Shape Parts
Feely Box—Shapes 2
The Shape of Things 2
Snapshots—Shapes
Small Group
Feely Box—Shapes 2
Building Shapes
Computer Center
Pizza Pizzazz 4
Piece Puzzler 2
Piece Puzzler Free Explore
Hands-on Math Center
Pattern Block Puzzles
Pattern Block Cutouts
Tangram Puzzles
1st Denotes first occurrence
Piece Puzzler 2
Children solve outline puzzles by putting together shapes and using tools to move the shapes into place. Children solve outline puzzles by putting together shapes and by composing pictures and designs.
In Piece Puzzler 2: Assemble Pieces, children can fill frames to form pictures in which each shape represents a unique role or function in the picture. Children have to drag, but sometimes also turn or flip, shapes into place. In addition, the shapes touch at their sides, encouraging children to see how shapes combine to fill regions. However, each shape's
more info

68


TRiad
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instruction
help
articles
BB Manage
index
credits
instruction
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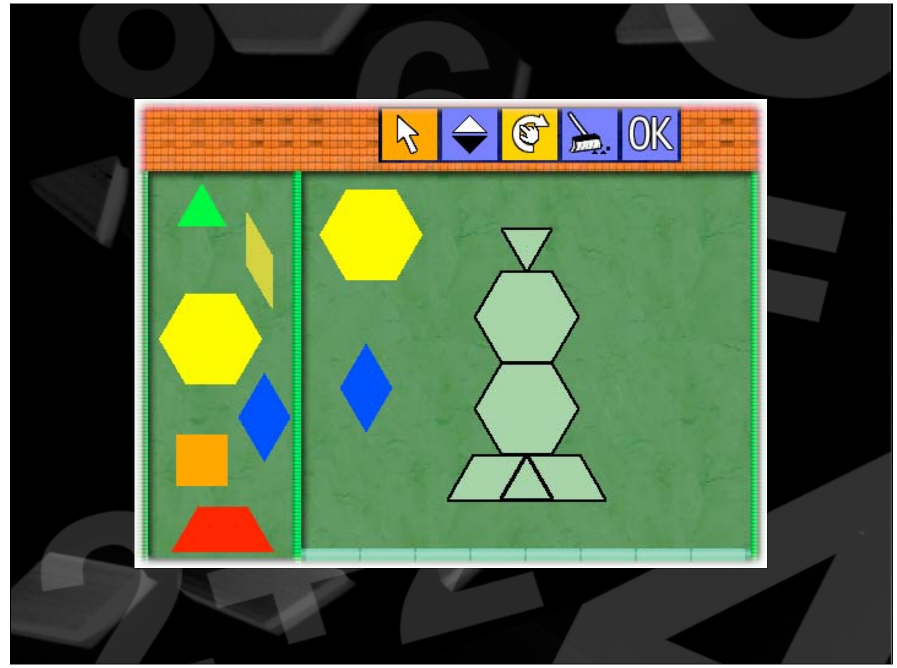
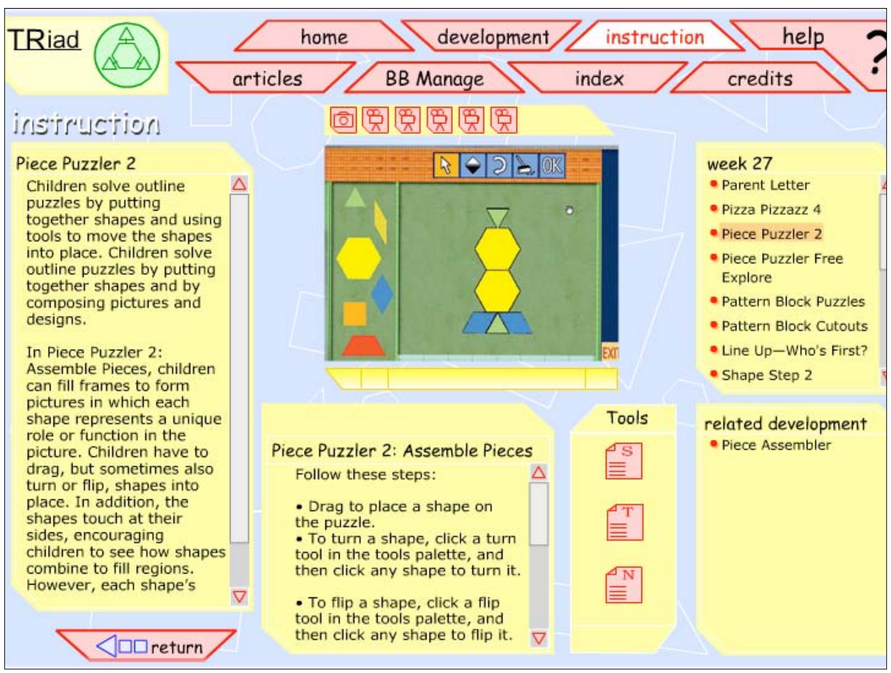
return
week 27
Parent Letter
Pizza Pizzazz 4
Piece Puzzler 2
Piece Puzzler Free Explore
Pattern Block Puzzles
Pattern Block Cutouts
Line Up—Who's First?
Shape Step 2
Piece Puzzler 2: Assemble Pieces
Follow these steps:
• Drag to place a shape on the puzzle.
• To turn a shape, click a turn tool in the tools palette, and then click any shape to turn it.
• To flip a shape, click a flip tool in the tools palette, and then click any shape to flip it.
Tools
related development
Piece Assembler

69


OK

70


TRiad
home
development
instruction
help
articles
BB Manage
index
credits
instruction
Piece Puzzler 2
Children solve outline puzzles by putting together shapes and using tools to move the shapes into place. Children solve outline puzzles by putting together shapes and by composing pictures and designs.
In Piece Puzzler 2: Assemble Pieces, children can fill frames to form pictures in which each shape represents a unique role or function in the picture. Children have to drag, but sometimes also turn or flip, shapes into place. In addition, the shapes touch at their sides, encouraging children to see how shapes combine to fill regions. However, each shape's
return
week 27
Parent Letter
Pizza Pizzazz 4
Piece Puzzler 2
Piece Puzzler Free Explore
Pattern Block Puzzles
Pattern Block Cutouts
Line Up—Who's First?
Shape Step 2
Piece Puzzler 2: Assemble Pieces
Follow these steps:
• Drag to place a shape on the puzzle.
• To turn a shape, click a turn tool in the tools palette, and then click any shape to turn it.
• To flip a shape, click a flip tool in the tools palette, and then click any shape to flip it.
Tools
related development
Piece Assembler

71


TRiad
home
development
instruction
help
articles
BB Manage
index
credits
development
Piece Assembler
Makes pictures in which each shape represents a unique role (e.g., one shape for each body part) and shapes touch. Fills simple outline puzzles using trial and error.
return
geometry
composing shapes
Pre-Composer
Pre-DeComposer
Piece Assembler
Picture Maker
Simple DeComposer
Shape Composer
Substitution Composer
Piece Assembler
This girl demonstrates her ability to assemble the blocks to fit the puzzle accurately with no gaps.
Tools
related instruction
Piece Puzzler 2
Puzzles
Pattern Block Puzzles
Pattern Block Cutouts
Tangram Puzzles
Tangram Pictures
Tangram Puzzles
Tangram Pictures (Introduction)

72

Teachers of Older Students

- Herb Gross' sites:
  www.adjectivenounmath.com
  www.lovemath.org
- http://www.wirelessgeneration.com/posters

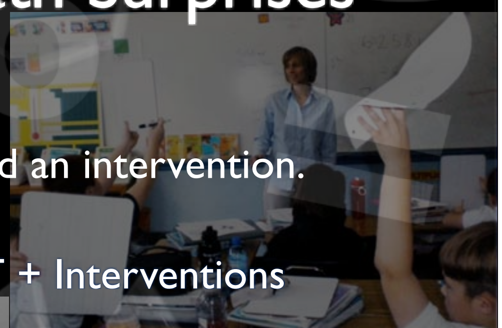
73

Early Math Surprises

1. Early math has surprising *predictor power.*
2. Young children have the potential to learn powerful math.
3. #2 is a surprise to most educators.

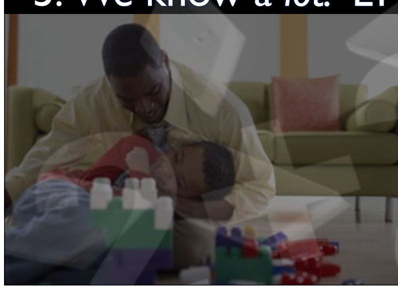
74

Early Math Surprises



4. Most children need an intervention.

5. We know a *lot.* LT + Interventions



75

76

77

78

References

- Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. NY: Routledge.
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COLÁISTE PHÁDRAIG
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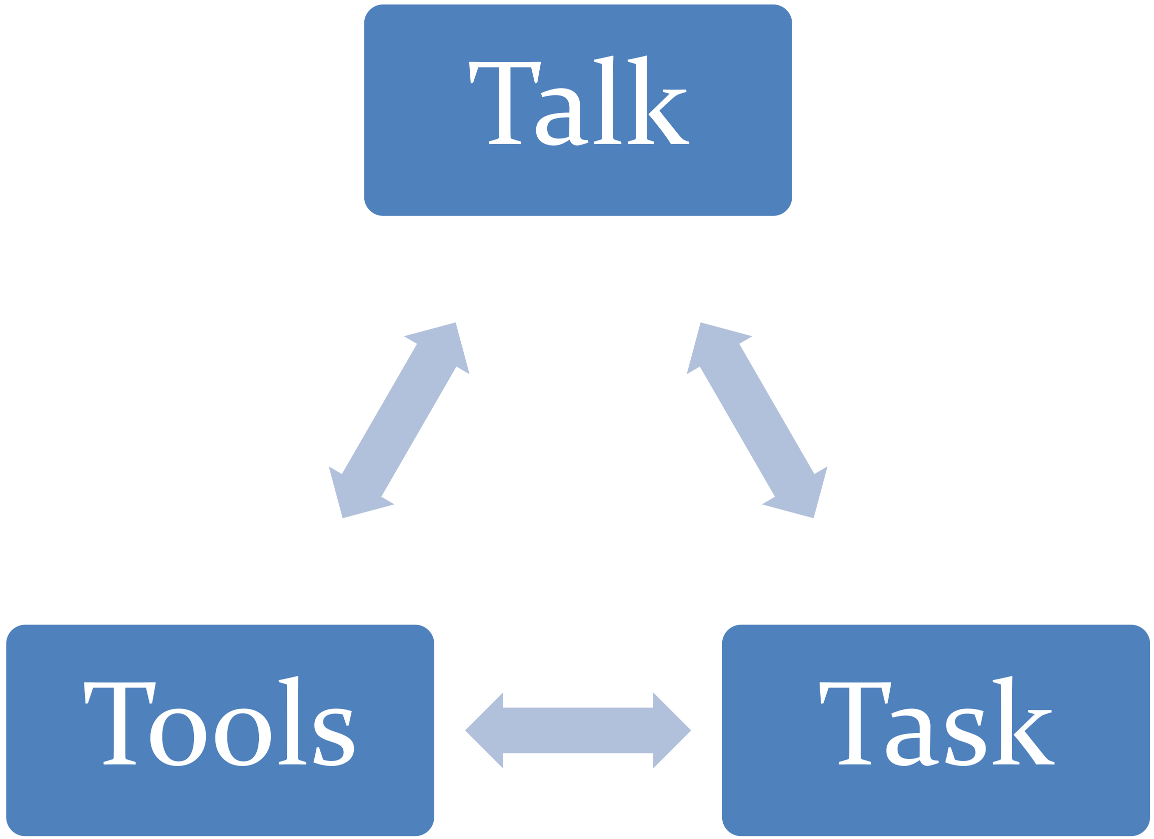
A College of
Dublin City University

Facilitating young children's understanding of the 'equal' sign

Dr Thérèse Dooley, St Patrick's College, Drumcondra and Aisling Kirwan, Holy Family National School, Rathcoole, Co. Dublin

www.spd.dcu.ie



Teaching mathematics well requires attention to three aspects: Tasks, Tools and Talk (Askew, 2012)

- Any teaching, including the particular case of mathematics, actually teaches far more than the content: children are learning much more than just mathematics in mathematics lessons. They are learning a lot about themselves, about their peers and their relationships.

(Askew, 2012, p.xvii)

Talk

- Children talking about their mathematical thinking is identified as an important way for them to make their thinking visible (Fuson et al., 2005)
- It involves encouraging and supporting children's communication, and their initial efforts to engage in reasoning and argumentation
- The teacher has a key role to play in providing a model of the language that is appropriate in a particular mathematical context
- Language as a tool for developing children's understanding of concepts, strategies and mathematical representations
- Challenge of eliciting talk about mathematics with young children not to be underestimated

(Report no. 18, Chapter 2)

Tasks

- Open-ended tasks support student thinking and exploration.
- Differentiation can be facilitated by providing the same basic task to all students and taking individual needs into account (e.g., extra supports, extension activities etc.).
- Productive task engagement requires that tasks are closely linked to a student's current level of knowledge and understanding but are 'just beyond' his or her cognitive reach.
- Tasks can remain cognitively challenging throughout a lesson if emphasis is placed on ways of thinking rather than on correct procedures, if sufficient time is allocated to completion of the task and if there is a continued emphasis by the teacher on justification and explanation.

(Anthony and Walshaw, 2007 (selected) in Report no. 18, Chapter 2)

Tools

- Learning environments that are rich in the use of a wide range of tools (including digital tools) support all children's mathematical learning.
- Tools – including both physical artefacts and symbolic resources – are an integral aspect of human cognition and activity.
- Among the forms of representation that children use to organise and convey their thinking are concrete manipulatives, mental models, symbolic notation, tables, graphs, number lines, stories, and drawings (Langrall et al., 2008).

(Report no. 18 – Chapters 2/3)

Cobb (2007) sees teaching as a coherent system rather than a set of discrete, interchangeable strategies. It encompasses four elements:

- a non-threatening classroom atmosphere
- instructional tasks
- tools and representations
- classroom discourse

(Report no. 18, Chapter 1)

Background and Context

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- The rigid view that first class children had of the 'equal' sign was evident in their attempts to solve missing addend equations.

E.g: 4 + __ = 10

- Participation in lesson study with my M.Ed group led to the planning and teaching of a lesson where Cuisenaire Rods were used to aid children to better understand the 'equal' sign.

Format of the research

- Research took the form of a 'Teaching Experiment' whereby the dual role of teacher-researcher was undertaken.
- Study took place in a mixed first class of 28 pupils.
- Fourteen lessons were carried out over a four week period.

Issues research attempted to address

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- Children's misinterpretation of the 'equal' sign as a 'do something' symbol (Frieman & Lee, 2004).
- Inability to develop a relational understanding of the 'equal' sign, that is, seeing the connections between both sides of the equation as a result of this misinterpretation (Warren, 2006).
- Misunderstanding of unconventional equations due to such a limited view of the 'equal' sign.

E.g: $6 = 3 + 3$, $10 = 10$, $7 - 1 = 5 + 1$, $3 = 1 + 1 + 1$

Data Collection

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- Audio-visual recordings
- Children's reflective journals
- Researcher's reflective journal
- Work samples
- Field notes
- Pre, post and delayed post tests

Why Cuisenaire Rods?

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- Simple, uncomplicated pieces of apparatus with a close relationship to number (Trivett, 1959).
- No research to suggest that Cuisenaire Rods had been previously used to explore the 'equal' sign.
- Weight balances had been used to some success, but problems arose in making the link with number (Warren, 2007).

Cuisenaire Rod Tasks

- **Task 1**: Find a combination of rods that are the same length as an orange rod.
- **Task 2**: Make a train equivalent in length to a train with the following combination of rods; light green, black, pink and dark green.
- **Task 3:** Make a train equivalent in length to a train with the following combination of rods; orange yellow and pink. Use at least four different rods. Now try it using as few rods as possible.
- **Task 4:** Investigate whether or not an orange rod (10) is equal to the following combination; dark green (6), pink (2) and pink (2).
- **Task 5**: Is 7 = 7? Investigate using the rods.
- **Task 6**: Is 6 = 1 + 1 + 1 + 1 + 1? Investigate using the rods.
- **Task 7:** Can you use the rods to find the missing addend for the following equation; 7 + 1 = 4 + ___

Use of Pre, Post and Delayed Post Tests

- Tests based on one carried out by Baroody and Ginsberg (1983) whereby children were asked to correct conventional and unconventional equations.
- Pre and post tests made up of the same format: the children were asked to decide if a number of equations were true or false and to give a rationale for their answer.
- Delayed post test administered a month after the teaching experiment (TE) and consisted of some true/false questions and missing addend equations for the children to solve.

Pre-Test Findings

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- 12 items on the test, one conventional equation in the form a + b = c, the other 11 were unconventional equations in the form a = b + c, a = b, a + b = c + d.
- 24 of the 26 children marked the conventional equation as being correct.
- The unconventional items were marked as incorrect by the majority of the class.
- False items such as 4 + 2 = 1 + 1 + 1 were marked as incorrect by the majority of children, but this was as a result of their structure being 'weird' or 'strange' rather than through understanding the relationship the 'equal' sign establishes between both sides of the equation.

3 = 2 + 1

- "It doesn't make sense"
- "It is false because the sum is sepose to be 2 + 1 = 3 or 1 + 2 = 3"
- "It is the rong way around"
- "You mixed it up its supposed to be 2 + 1 = 3"
- "It is false because it is odd and weird"
- "The equal sign can not go in the middle"

Focus Group

- Six children, two higher achieving pupils, two middle achieving pupils and two lower achieving pupils.
- Following the pre-test, it was established that all the children in the focus group had a 'do something' view of the 'equal' sign.
- The children had difficulty interpreting the unconventional equations as a result of this misunderstanding.

Lessons 1 - 4

- Focused on equivalence and allowing children time to explore the properties of the rods – colour, length etc.
- Tasks given whereby children had to find equivalent rods to match a variety of rod combinations.
- Tasks became progressively more difficult and children had to build 'trains' of rods of specific lengths.

Lessons 5 - 8

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- Focused on recording the rod combinations, firstly on blank paper to allow the children to concentrate solely on recording the rods without having the additional focus of recording numbers.
- After initial recordings, rod combinations were recorded on squared paper, with the association of the rods with number being the focus.
- Children began to record sentences to describe the rod combinations initially, with the colour names of the rods eventually being replaced by their respective numbers. E.g: white – 1, red – 2 etc.

Initial rod recordings on blank paper 1

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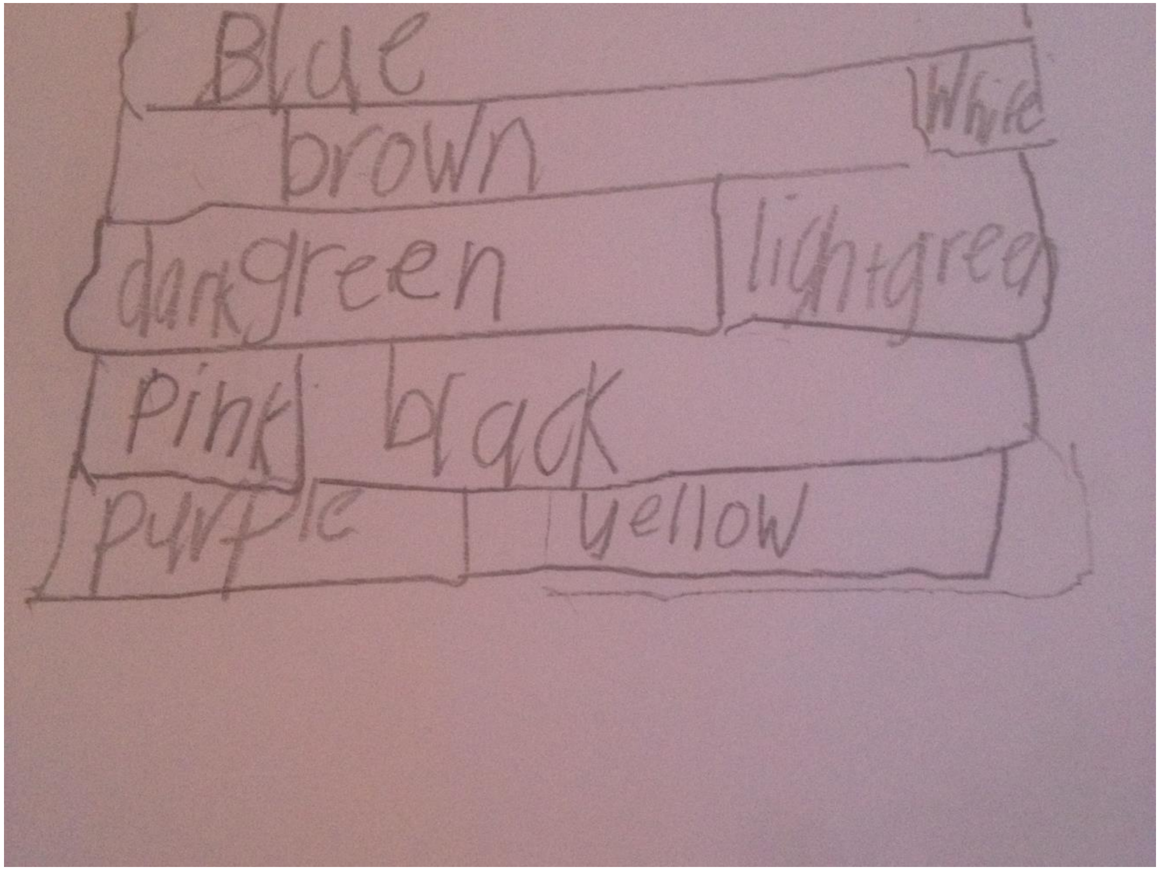
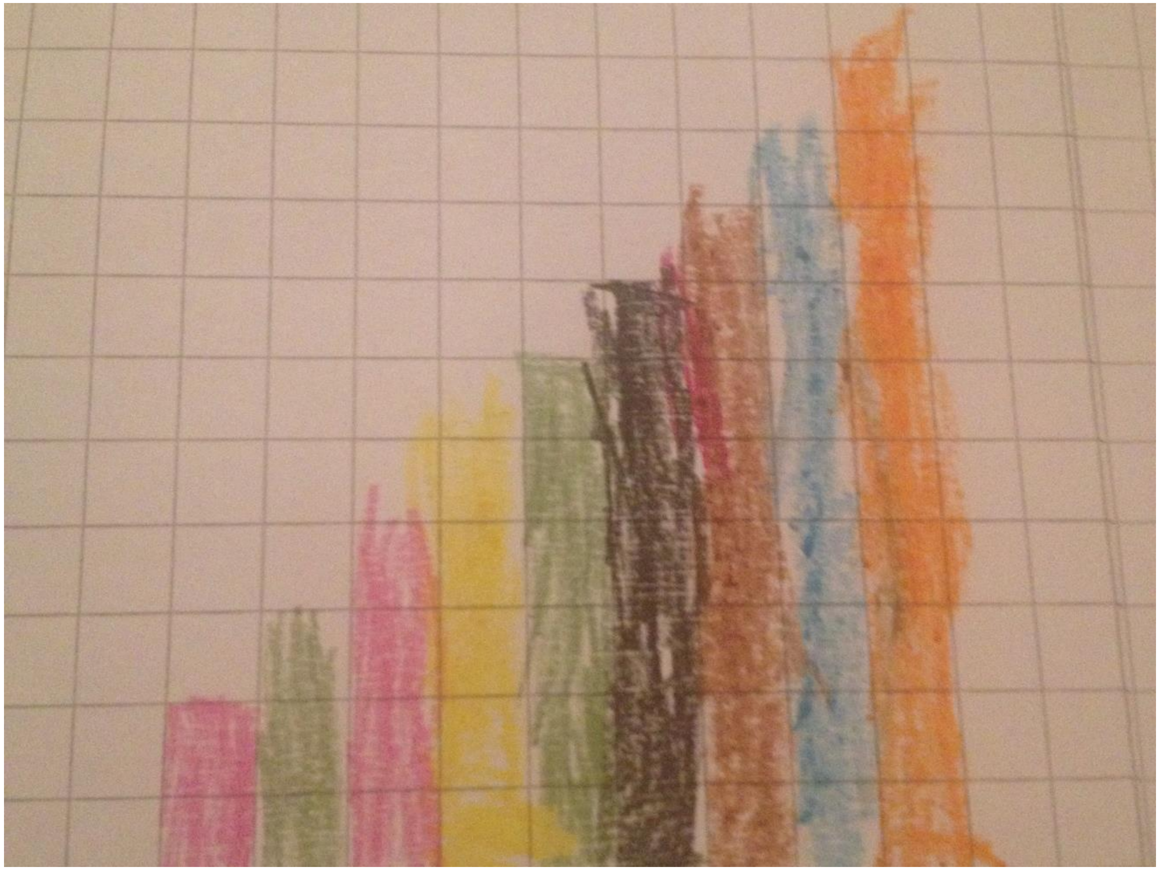
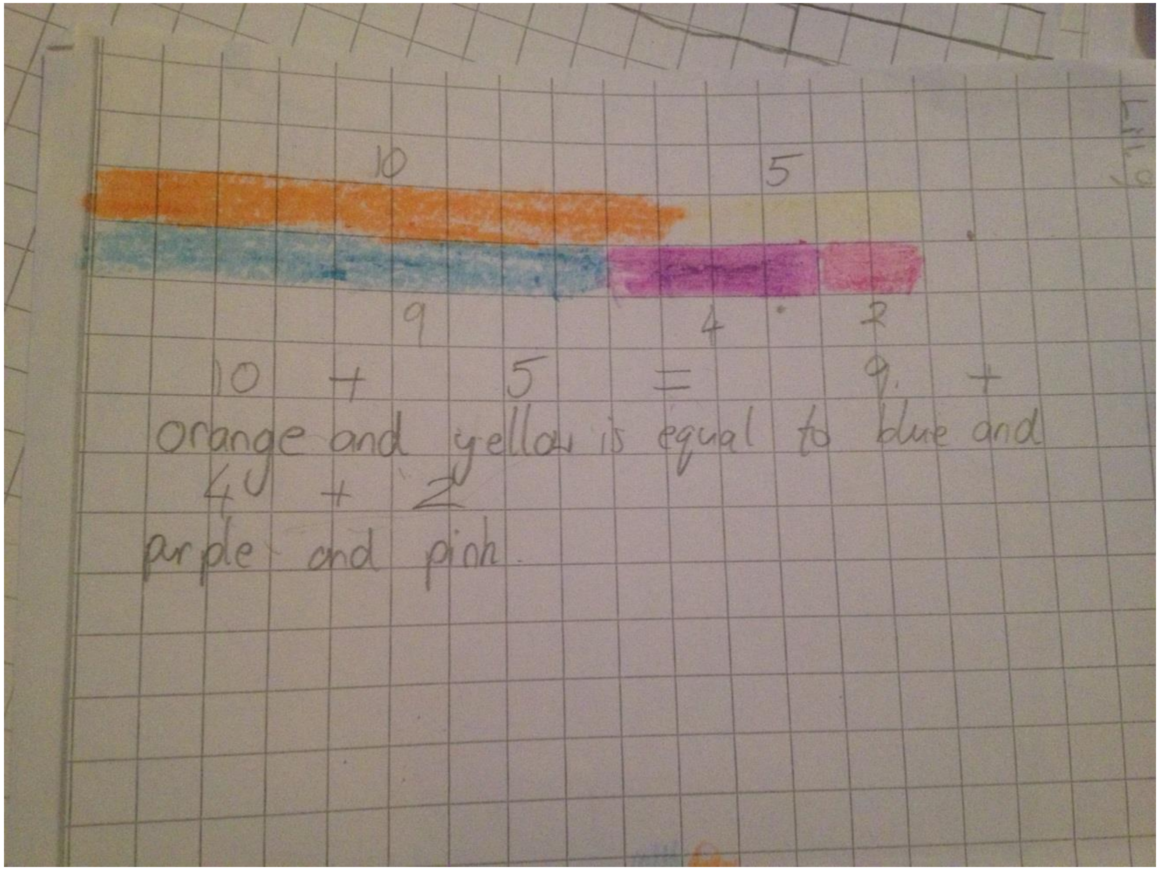


Initial rod recordings on blank paper 2

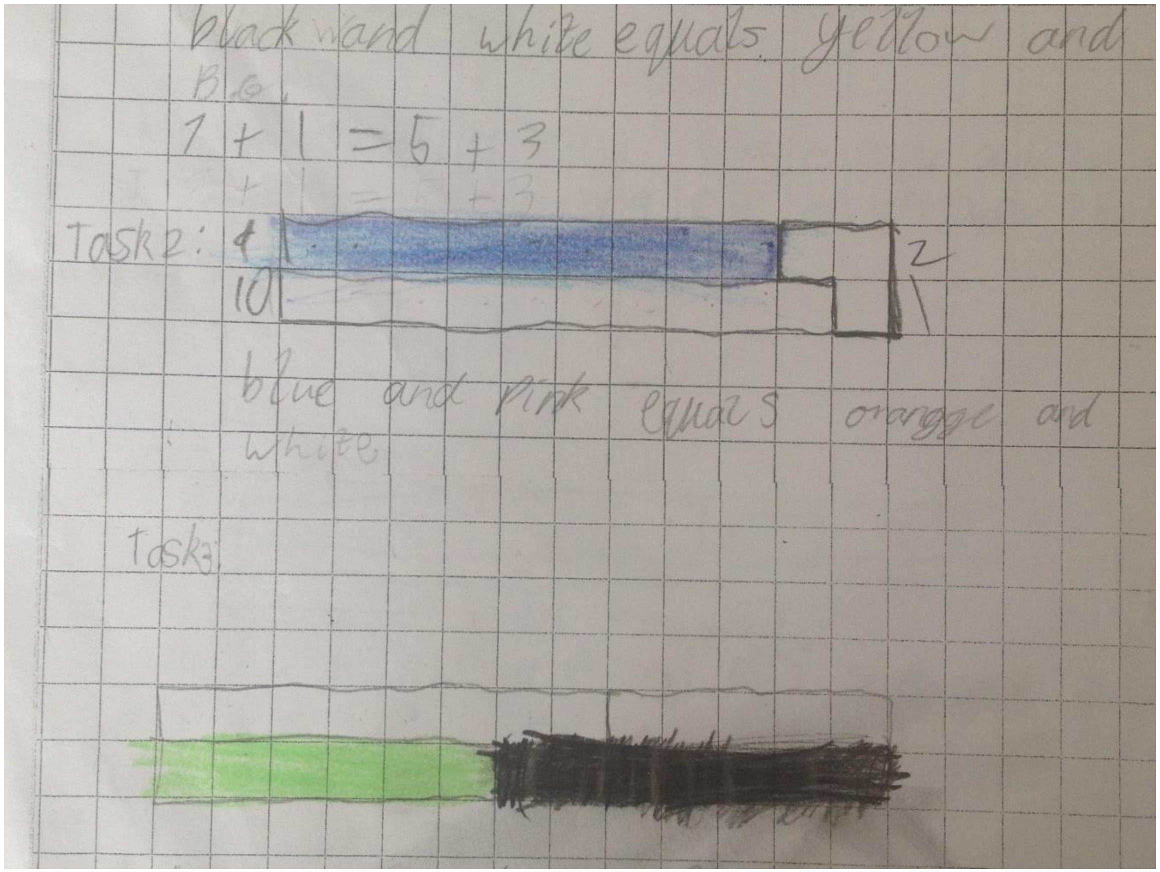
Initial recordings on squared paper



Recording rod combinations with words, numbers and the ‘equal’ sign



Blue (9) and pink (2) equals orange (10) and white (1)



Yellow is equal to a white and a white and a white and a white and a white.

5 = 1 + 1 + 1 + 1 + 1

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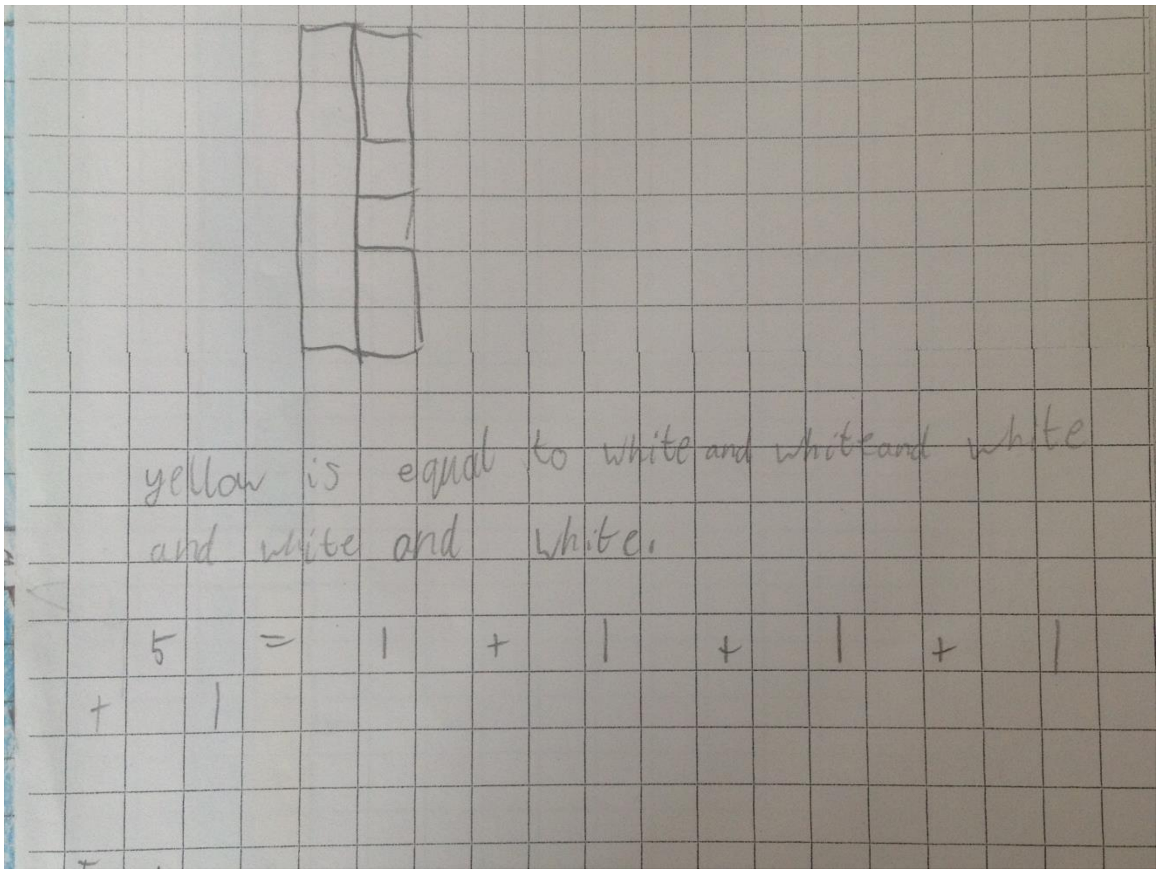
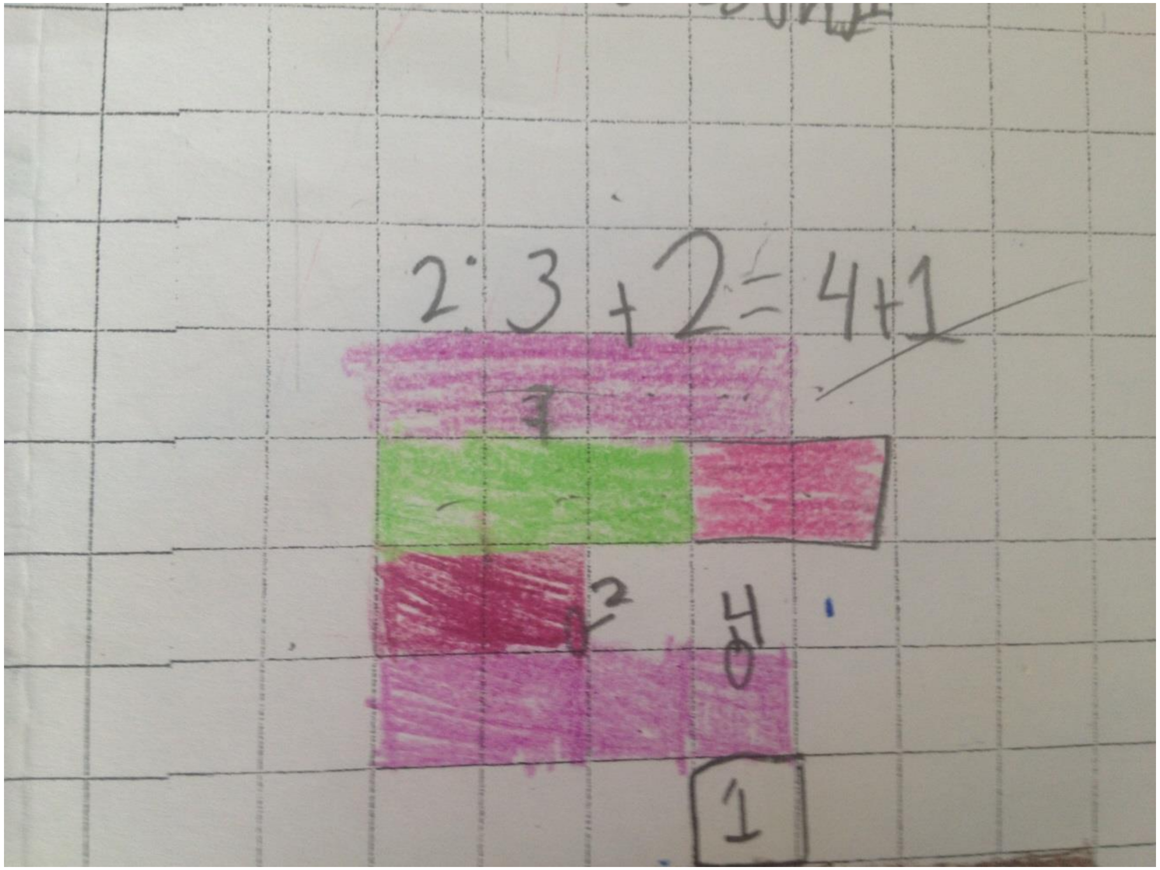


Lessons 9 - 11

- Focused on examining a variety of unconventional equations using the rods, as well as missing addends.
- Aimed at allowing the children to make a deeper connection between the rods and number.
- The 'equal' sign indicated the relationship between equivalent combinations of rods.

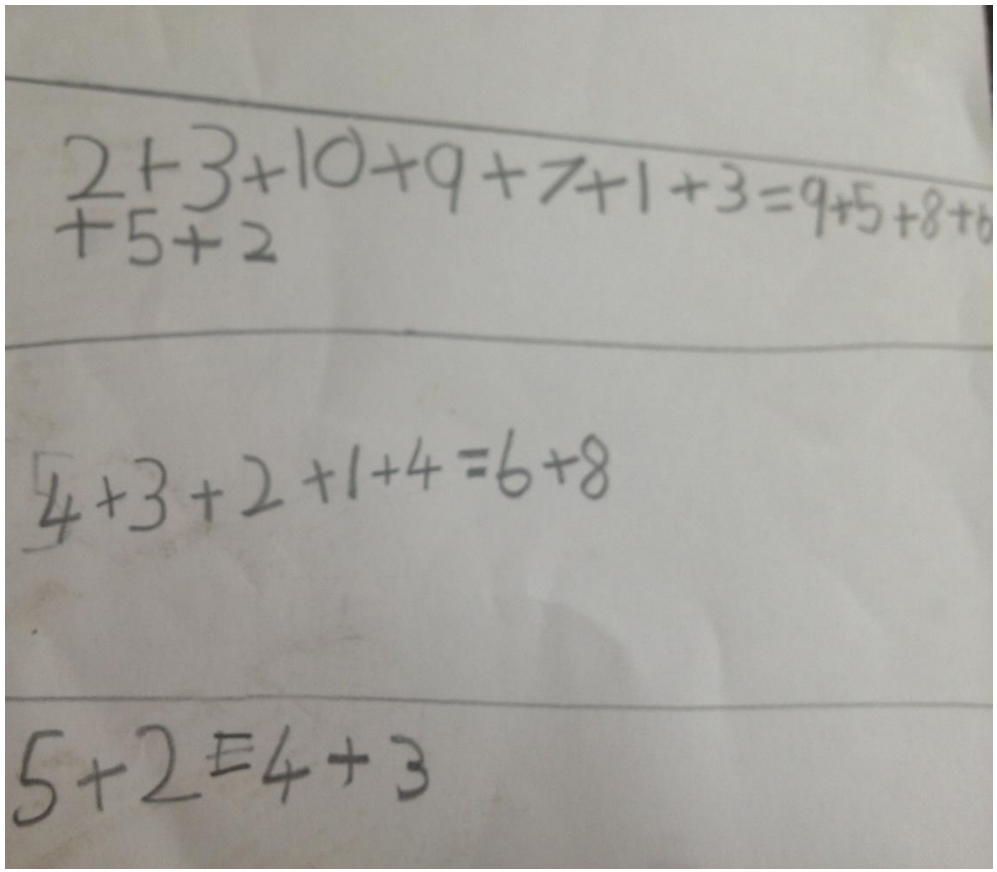
Missing addend: 3 + 2 = 4 + ___



Lessons 12 - 14

- Focus on moving the children away from the rods and put the emphasis on number.
- The children worked through correcting various unconventional equations without the rods.
- Continued to solve for missing addend by focusing on the relationship the 'equal' sign made between both sides of the equation.
- Children invented balanced equations of their own.

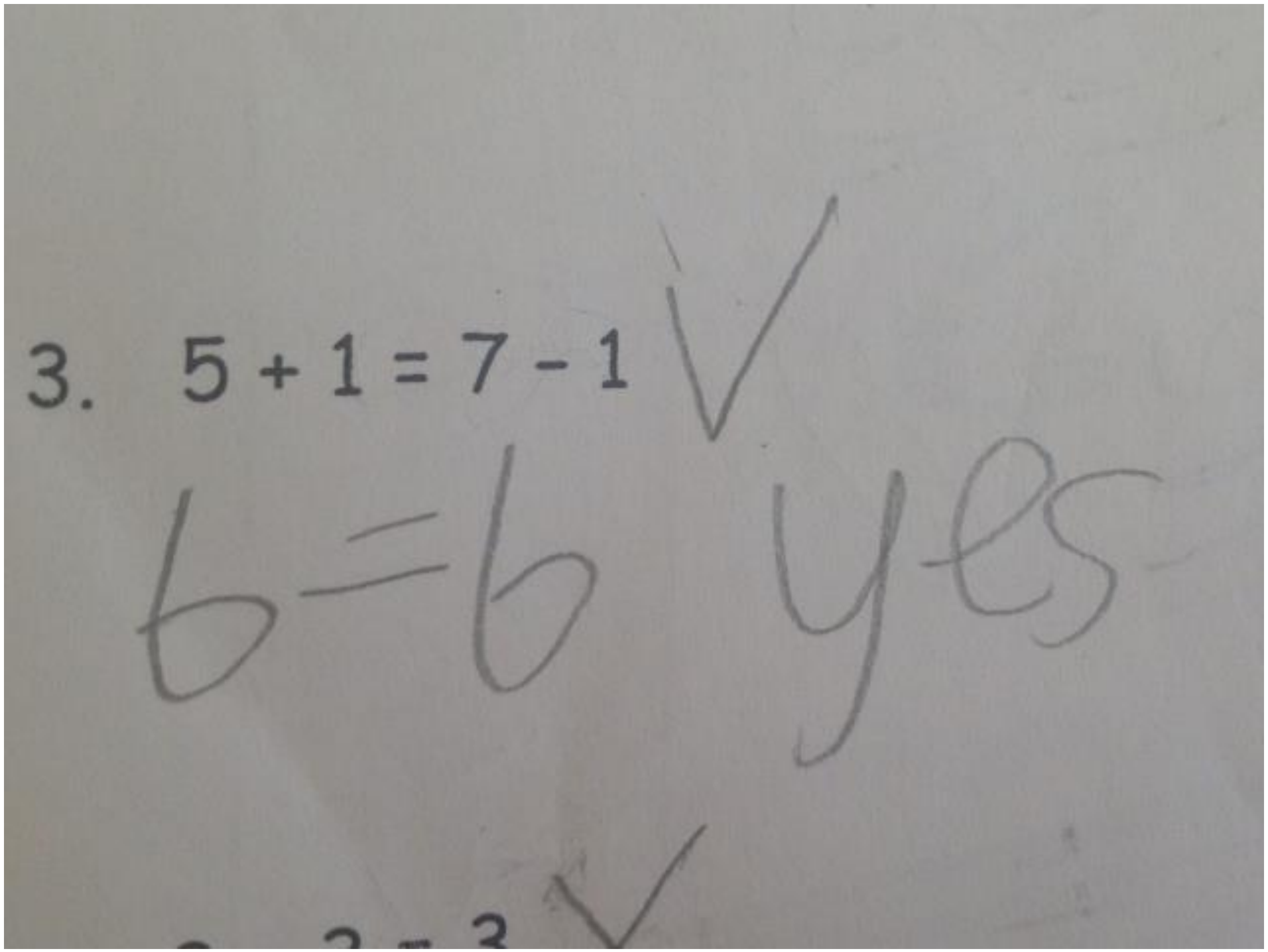
Children's invented equations: Making the move from concrete to symbol



Results of Post-Test

- Significant improvement overall in results from pre to post-test.
- Some children started to show that they had made the transition from concrete to symbol, and had begun to develop a relational understanding of the 'equal' sign.
- Important to note that not all children made such a transition, and still had confusion around the 'equal' sign.

Balancing both sides of the equation



Showing an understanding of the relationship the ‘equal’ sign makes between both sides of the equation.

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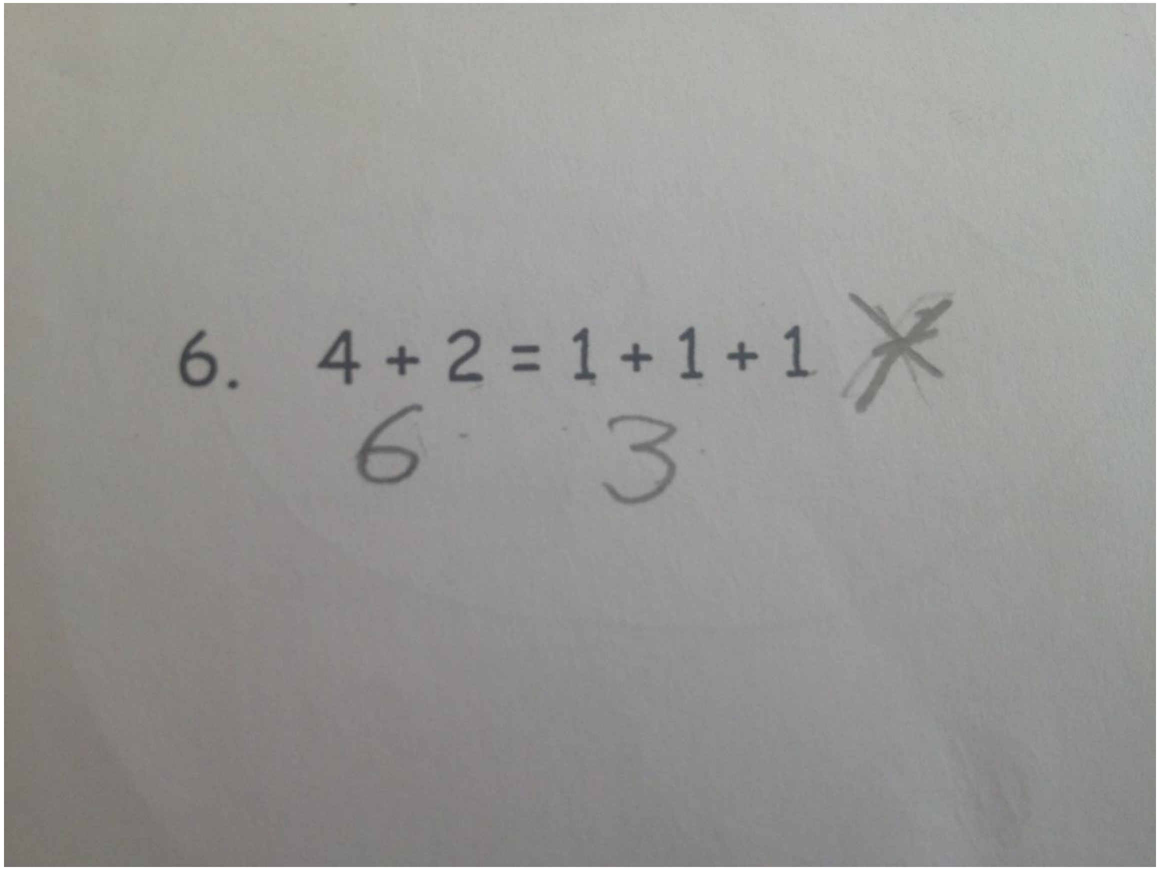

Transition from concrete to symbol not yet made

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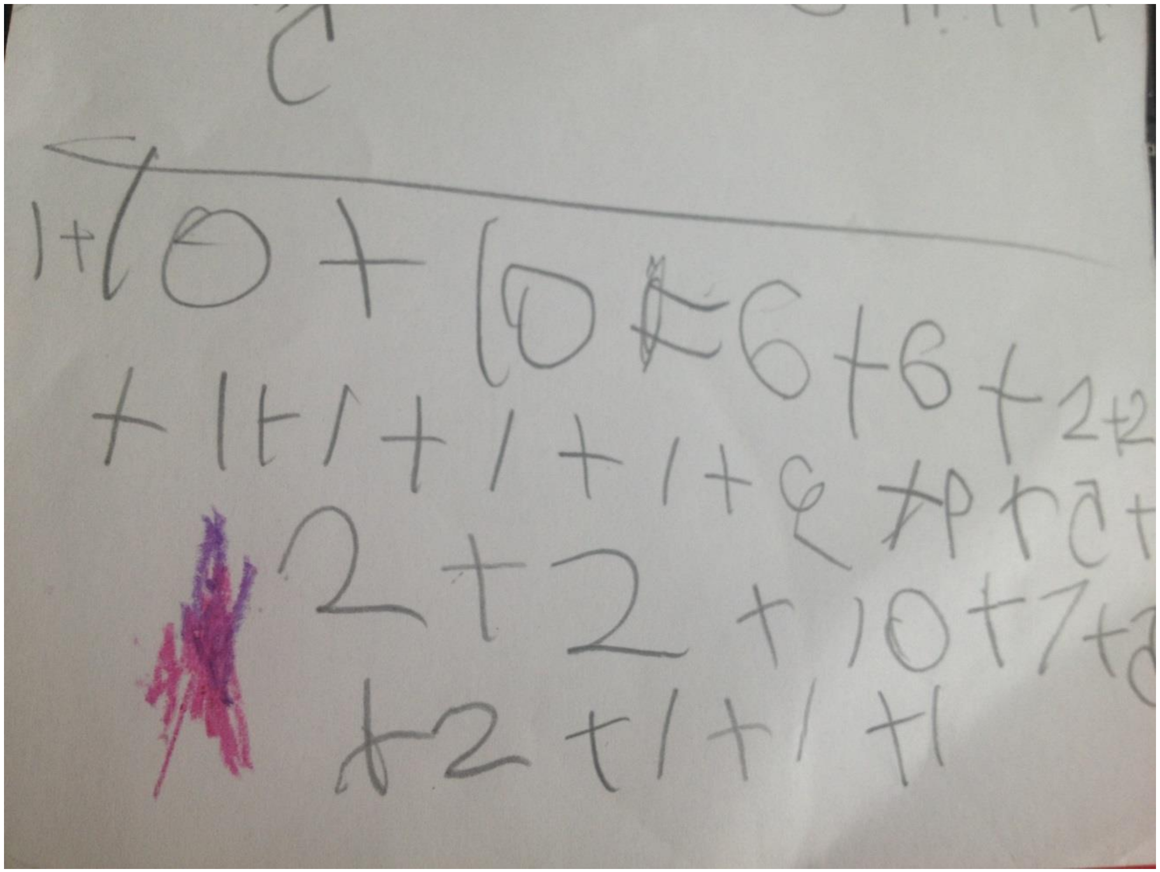

Post-Test: 3 = 2 + 1

- "It is true because it is just backwards"
- "There is 3 = 2 + 1 and that is 3 = 3"
- "3 is equeal to 2 + 1 because three is equeal to 3"
- "It is true it is just backwards"
- "It is like 2 + 1 = 3"
- "Equal because a white and a pink is equal to a light green"

Delayed Post-Test

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- Took place a month after the initial post-test.
- Very little change from the results of the post-test.
- Most children seemed to show that transfer of learning had taken place.
- The majority of children successfully solved the missing addend section which was not included in the post-test.

Concluding Findings

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- A lot of children developed a more relational view of the 'equal' sign after the teaching experiment.
- Children became more accepting of unconventional equations, and began to invent their own ones.
- Most children could successfully balance equations after the teaching experiment.
- The rods were used as referents by the children, especially when solving missing addend problems.
- Some children made the move from concrete to symbol, and were better able to read and interpret equations as a result. Not all children made this transition, and some were not as yet clear in their understanding of the 'equal' sign.

Limitations of the Study

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- Small scale teaching experiment
- Study conducted over a short period of time, therefore there is scope for further development and consolidation of learning if this was to be conducted over a longer timeframe.
- There is further scope to extend children's understanding of the 'equal' sign to include 'as a substitute for' as well as understanding the need to balance both sides of the equation.

Implications

- This teaching experiment could be implemented at a more basic level for the junior classes before the formal introduction to symbol.
- A focus on understanding was paramount to this TE, and moving the focus away from the textbook was highlighted as a result.
- The use of appropriate manipulatives in the classroom is recommended to aid children to move from concrete to symbol. Such developments do not occur in a linear fashion, nor do all children make the move.

Early Learning Initiative

National College of Ireland

Sharing Learning through the NEYAI Docklands Early Numeracy Programme

Supporting parents, communities and schools in the education of children

Early Learning Initiative
National College of Ireland

Working in partnership with local communities to support educational journeys and achievements

- Address educational disadvantage and its impact on personal and career development
- Provide a range of innovative support programmes for children and their parents from early years to third level
- Uses community action research (Plan, Do, Review) to implement national policy and programmes

Collaborative Projects

Parent Child Home Programme

NEYAI Early Numeracy Programme

Language

Literacy

Numeracy

Educational & Career Guidance

Second & Third Level

Parenting Programmes

Primary

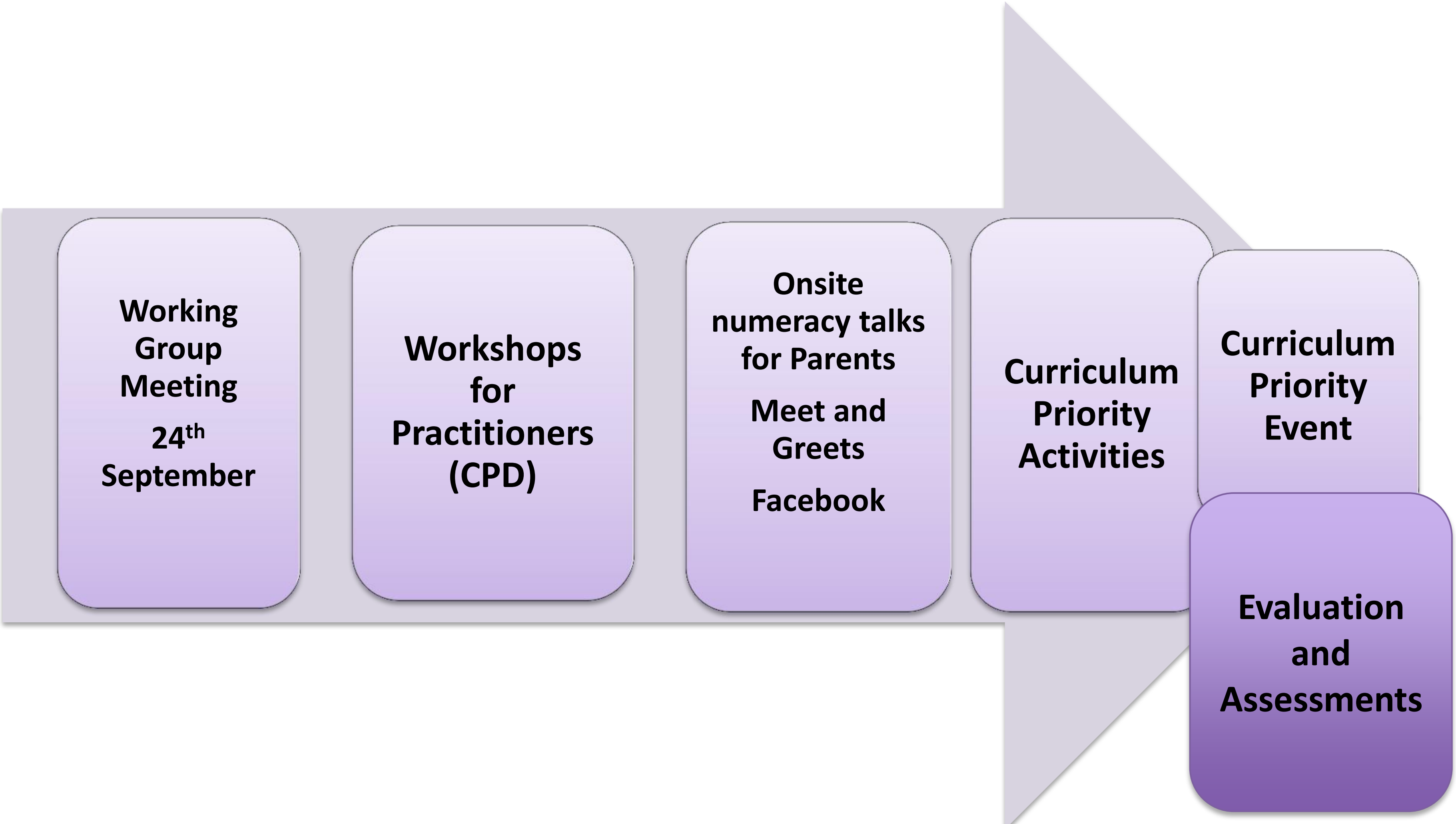
Rationale

- Internal ELI evaluations highlighted the low levels of numeracy in the Docklands area as well as the lack of support for parents in Mathematics (ELI 2010).
- National and international reports (DES 2005a; Surgenor et al 2006; Shiel et al 2007; Eivers et al 2010) emphasised how young people in Ireland were poorly prepared for future Mathematical needs as students and citizens.
- International literature review revealed
  - Opportunities for pre-schoolers to learn mathematics are often very inadequate (National Academy of Sciences 2009).
  - Enormous differences in the mathematical knowledge of children when they begin school (US Maths Recovery Council 2005; Northwestern University 2007; Every Child a Chance Trust 2009)
  - Those that are among the least advanced of their class remain so throughout their schooling and often give up on Mathematics.
- The lack of proficiency in maths-based subjects can be the trigger for non-completion at third level (HEA 2010).

Docklands Early Numeracy Project Objectives

- To improve the educational outcomes for children in the Docklands in numeracy
- To increase parental involvement in their children's development, learning and education by providing a variety of on-going supports for parents of young children.
- To support early childhood care and education workforce in implementing *Aistear, the Early Childhood Curriculum Framework* and *Síolta, The Quality Framework for Early Childhood Education*
- To ensure continuity and progression in Mathematical learning for children moving from home to early years settings to the local schools

Community Action Research
Usual Schedule of Events



NEYAI Numeracy Curriculum Objectives

(Taken from *Aistear*)

	Babies (0-18 months)	Toddlers (12 months – 3 years)	Young Children (2½ -6 years)
Communication	Watches, listens and responds to adults when they use Mathematical language	Responds to and understands Mathematical language in everyday situations (p. 38)	• Develop basic counting skills (1-10) • Develop an understanding of the meaning and use of numbers in their environment • Understands and uses positional language such as up, down, out, behind
Exploring and Thinking	Experience and begin to understand simple cause and effect Develop the concept of object permanence	Compare, sort, categorise and order things Develop a sense of time, shape, size, space and place	• Classify, sequence, sort, match, look for and create patterns and shapes • Develop an understanding of concepts like measures (weight, height, volume, money, time) • Use mathematical symbols to give and record information, to describe and make sense of their own and others experience • Develop higher-order thinking skills such as problem-solving, predicting, analysing, questioning and justifying

Themes for 2014-2017

	Year 1 2014-2015	Year 2 2015-2016	Year 3 2016-2017
Term 1	Positional Language	Time	Money
Term 2	Counting	Measurement	Number
Term 3	Shapes	Sequence and Pattern	Symbols in the Environment

NEYAI Docklands Early Numeracy Working Group

- Chosen for your interest in the project and in early numeracy as well as your ability to lead the project in their setting.
- Meet 4 times a year approx. (June, September and December 2011)
- Communicating between services and working group (open, honest, critical, responsible)
- Responsible for developing, planning and implementing the programme at front-line service delivery level using the community action research process

Role of the Working Group

- Bring and share early numeracy expertise and experience from working with various age-groups
- Network with other settings
- Input re. Theme/ Focus of Curriculum Priority Week
- Input re. Resources, Activities etc.
- Communicate info. to their staff team
- Involve wider community

Co-ordinate curriculum priority week & related events in school/setting e.g. awareness of existing resources to complement c.p. wk activities

Docklands Early Numeracy Activity Week

Positional Language: on top, under, beside, in front of, next to, fit, inside, outside, between, around, on, off, into, out of, far, near, close to

Parents try this at home with your child……

Babies- (0-18months)

- Play "a where's teddy gone?" game and hide different objects as part of the game. Hide teddy under the blanket, behind your back, under your arm. Talk through what you are doing "I wonder where teddy is gone, is he under the blanket….?"

Observe- How does your child react? Smiles, laughs, follows you with their eyes, looks for teddy?

Wobblers/toddlers (12months-3 years)

- Play hide and seek with the child hiding first and you looking for them. Make sure to talk through what you are doing "Where's Lilly gone? Is she behind the door, Is she under the bed?.... Swap over and let your child find you.

Observe: How does your child react? Laughs, is having fun, copies you and uses positional language?

Young Children (2 ½ -6 years)

- Choose an item to hide (e.g. child's favourite toy) and give your child directions to find it (see positional language above). Swap over and get your child to hide the item and give you directions to find it.

Observe: How does your child react? Responds to your directions and find the item? Has difficulty following directions (in this case, make the directions simpler)?

NEYAI is funded by Atlantic Philanthropies, Mount Street Club Trust, The Department of Education and Skills and the Department of Children and Youth Affairs

First card – Changes

A4 rather than A5

Different cards for different age groups

Emphasise conversations

Categorising Play p. 54

Space for recording experiences & thoughts

Signed

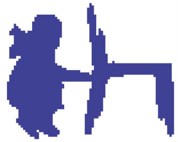
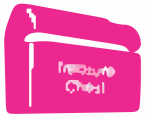
Docklands Early Numeracy Activity Week- Young Children- Pre-schoolers/Reception/Junior Infants

Words to use: Up, down, front, back, in, out, on top, over, under, beside, behind, in front of, next to, fit, inside, outside, between, around, on, off, into, out of, far, near, close to, forward, backward, across, left and right.

Child's Name:

Positional / Directional Language

Using positional / directional language when playing, singing and talking with your child, will help them to learn these important early numeracy concepts. Try these activities each day with your child and don't forget the rhymes at the back of the card.

	Monday	Tuesday	Wednesday	Thursday
				
Parent's Signature				
Play the games with your child, and talk about what you are both doing using the positional/ directional language	**Hide and Seek** Play Hide and Seek with your child hiding first and you looking for them. Talk through what you are doing, "where's Lily gone, is she **behind** the curtain, **under** the table, **beside** the couch? Swap over and let your child find you.	**Treasure Hunt** Choose an item to hide (e.g. child's favourite toy) and give your child hints to find it, using the positional language listed at top of card. Swap over and get your child to hide the item and give you directions to find it.	**I spy** Play the game 'I spy' and give your child clues about the item using positional language, e.g. it is **Next To** the mirror, **Under** the bed, **Inside** the shed. Swap over and let your child give you clues and practice their positional language.	**Obstacle Course** Set up a small obstacle course in the house, using cushions, chair, stool, table, blanket and anything else you can think of. Give your child directions, "Jump **over** the cushion, Go **under** the table, Go **around** the chair" If you are brave enough, swap over and let your child guide you!
Do you think your child enjoyed each activity? Please tick a smiley face.	☺ 😐 ●	☺ 😐 ●	☺ 😐 ☹	● 😐 ☹

Friday	
Teacher, can I please have a sticker?	

The Area Based Childhood Programme 2013-2016

Early Learning Initiative — National College of Ireland | Docklands Early Numeracy Project

Docklands Early Numeracy Activity Week - Pre-schoolers/Reception Class/Junior Infants

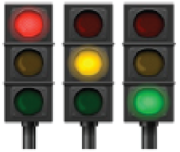
Words to use: sign, traffic lights, lollipop man, map, shape, stop!, straight, arrow, turn, right, left, follow, bus stop, rain, wind, sunny, toilets, chemist

Child's Name:

Symbol hunt!

Looking for and talking about symbols around you helps children's early numeracy. Try this activity each day with your child

Monday

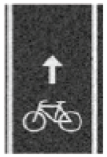


On your way home from school find a traffic light and draw it when you get home

Parents Signature

Teacher can I have a sticker?

Tuesday



On your way home from school find a cycle lane and draw it when you get home

Parents Signature

Teacher can I have a sticker?

Dear Parent:
What do you think your child has learned from this activity?
New Language ☐ starting to recognise symbols ☐ Other:______________________________

The Hokey Pokey
You put your right hand in, You put your right hand out, You put your right hand in, And you shake it all about, You do the hokey pokey and you turn yourself around thats what it's all about. Oooooh hokey pokey pokey, Oooooh hokey pokey pokey, Oooooooo hokey pokey pokey and that's what it's all about!- **repeat for** left hand, right foot, left foot, head, whole self

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NEYAI National Early Years Access Initiative
Promoting Better Outcomes for Children & Families

Early Learning Initiative
National College of Ireland

Docklands Early Numeracy Project

Docklands Early Numeracy Activity Week

Shape and Space- Key Vocabulary: square, circle, triangle, rectangle, round, curved, straight, corner, flat, edges

ECCE Practitioners:

ECCE Practitioners- try an activity with the children every day this week, use rhymes & songs to support and extend the activities (overleaf).

Babies- (birth-18months)

- Sing shape rhymes to baby during care routines, tracing the shapes in the song on their palms, tummy etc.
- When carrying babies around the room/ setting, point out different shapes and say the name aloud. If possible, allow baby to touch these and enjoy their different textures and shapes.
- Allow children to experiment with, and understand how different shapes and sizes will fit into spaces e.g. stacking cups, nesting boxes or dolls, cardboard boxes of various sizes.
- Treasure basket- during quiet time, where you can be close and attentive, allow baby to explore the shape, form and texture of the objects in the treasure basket

Assessment- How does the child react? Responds with interest? Engages with the objects? Is willing to explore and experiment?

Wobblers/Toddlers (12months-3 years)

- Messy Play/ sensory play – adult models tracing of various lines (straight, curved lines, zig-zag lines) in shaving foam, wet and dry sand, gloop, dried pasta, rice, etc. Children have an opportunity to observe the adult closely and copy the designs if they wish. Adult then progresses to making shapes such as circles, saying the word circle as they make the curved line join.
- Play dough time- make some play dough with the children- this provides lots of rich opportunities for language development. Using cutters or using plastic knives, cut out shapes - adult models use of language such as straight line, curved etc.
- Printing Time- make a collection of similar shaped items e.g. bottle tops, jam-jar lids, plastic plates. Using paint, encourage children to make prints with these objects and discuss the size and shape of each print

Assessment: Does the child copy your actions? Make their own shapes? Is he/she beginning to notice simple shapes and patterns in pictures? Does he/she begin to talk about the shapes of everyday objects?

Young Children (2 ½ -6 years)

- Fun with shapes- make pictures using pre- cut shapes (paper, foam, card)- e.g. two circles, one large and one small could make a snowman, use squares and rectangles to make a robot. Display the pictures for all to see!
- Chalk Shapes- bring the children outside to an area where an adult has used chalk on the ground to draw large versions of lines and shapes (e.g. square, circle, triangle). Children can then walk, run, hop, skip around and into these shapes. When children are familiar with this activity, they can draw the lines and shapes themselves.
- The 'Feely Bag' game- adult places a number of plastic shapes into a cloth bag. Children take turns putting their hand into the bag, picking a shape and describing it aloud. When they have finished doing this they can guess the name of the shape. When introducing this game, spend plenty of time modelling the language and skills required to play.
- Construction play- play with blocks or open-ended construction materials (e.g. cardboard boxes) allows for many opportunities to explore shape and space. Try setting up a construction corner, providing a range of materials of various shapes, sizes, textures and weights. Try offering a mix of free-play and structured activities, spend plenty of time modelling the language and skills required to extend and enhance the learning experience.

Assessment: Does the child show an interest in shape and space by playing with shapes and making arrangements with shapes for example through engaging in construction activities or by talking about shapes or arrangements? Does he/she show awareness of similarities in shapes in the environment? Does the child show curiosity about and observation of shapes by talking about how they are the same and different

NEYAI is funded by Atlantic Philanthropies, Mount Street Club Trust, The Department of Education and Skills and the Department of Children and Youth Affairs

Docklands Early Numeracy Activity Week

Positional/Directional Language - Key Vocabulary: Up, down, front, back, in, out, on top, over, under, beside, behind, in front of, next to, fit, inside, outside, between, around, on, off, into, out of, far, near, close to, forward, backward, across, left and right.

ECCE Practitioners - try an activity with the children every day this week, use rhymes & songs (overleaf) and your own ideas to support and extend the activities.

Babies (birth-18months)

- Play Peek-a-boo, hiding your face behind your hands and taking them away slowly. Talk through what you are doing, "Where's Mary gone, Behind here..."? You can also encourage the children to hide their own face behind their hand and describe what they are doing, "Where is Jake?".
- Play a game of "where's Teddy gone?" and hide different objects as part of the game. Hide teddy under the blanket, behind your back, under your arm. Talk through what you are doing "I wonder where teddy is gone, is he under the blanket….?"
- Give babies lots of opportunities to experience spatial awareness, letting them crawl or climb over cushions, under tables, in and out of boxes, through tunnels, inside tents and behind curtains, where possible describing to them what they are doing and giving them the language.
- Read the book "Where's Spot". Emphasise the positional/directional language e.g. behind the door, inside the clock, and give the children the chance to pull up the flaps. Use the toys in the baby room to make the story come to life.
- Read the book "Slow Snail". Emphasise the positional/directional language in the story e.g down her flower, over a stone. Trace your finger along the snail's path and encourage the children to do it also, while you describe what they are doing.
- Use lots of positional/directional language when babies are playing with bricks, blocks, stacking cups or stacking ring, describing what they or you are doing, "Look, we placed the yellow brick on top of the blue brick, the red cup inside the black cup..etc"
- Use lots of positional/directional language when babies are engaged in physical and outdoor play, "Look at you going up the slide, rolling under the table, climbing on top of the mat, crawling inside the tent, going behind the cupboard..etc"
- Emphasise positional/directional language during everyday activities such as changing and feeding time, "We'll get you up on top of the changing table, take your nappy off, find the cream inside the box..", "Let's all get our cups on the table"

Assessment: How does the child react? Engages with the games and activities? Does he/she begin to understand and recognise positional/directional language?

Wobblers/Toddlers (12 months-3 years)

- Emphasise positional/directional language when toddlers are engaged in physical and outdoor play, such as; "Look at you under the table, on top of the slide, under the tree..etc".
- Play Hide and Seek with the child hiding first and you looking for them. Make sure to talk through what you are doing "Where's Lily gone? Is she behind the door, is she under the table?"
- Play Treasure Hunt Game- while looking for an object hidden by the adult, model positional language e.g. "I wonder where teddy is gone, is he under the chair, let's look under here?" "Will we look up here?"
- Set up a Small World Play Scene e.g. Garage, Doll's House, Farm, Happyland School bus or Ikea Playmat. Talk to the children about where to put certain objects. Take turns with child: they tell you where to put an object, then vice versa.
- After a game/activity, use the traffic light poster to prepare toddlers for quiet time/story time.
- Read the book "Where's Spot". Emphasise the positional/directional language e.g. behind the door, inside the clock, and encourage the toddlers to pull up the flaps and talk about what they see. Use the toys you have in the room to make the story come to life.
- Read the book "Slow Snail". Emphasise the positional/directional language in the story e.g down her flower, over a stone. Encourage the toddlers to trace their fingers along the snail's path and talk about the story. Make a connection to outdoor play and the natural environment and collect props in the story such as flowers, stones and sticks. Watch out for snails in the garden or on trips to the park.
- Read the book "We're Going on Bear Hunt". Describe the journey and how they go through, around, under, over each obstacle. Toddlers will love acting out the journey themselves around the room – using positional/directional language as directed. This book has great rhythm and can also be turned into a song. Encourage the toddlers to make pictures based on the books they have read.
- Model and encourage the use of lots of positional/ directional language when playing with bricks, blocks, stacking cups or stacking ring, and pegboard, describing what they or you are doing, "Look, we placed the yellow brick on top of the blue brick, the red cup inside the black cup..etc"
- Model the use of lots of positional/directional language during sensory play (sand, water, gloop, shaving foam, play-dough)
- Take the children on a visit to the local library and look for books that focus on positional/directional language and read them to the children. This could be a nice event to get parents involved with.

Assessment: How does the child react? Does he/she begin to understand and recognise positional/directional language? Does he/she begin to use positional/directional language?"

Young Children (2 ½ -6 years)

- Play a game of "Simon Says" using an object like a beanbag. Each child gets a beanbag- adult gives directions as to where to place the beanbag- e.g. "Please put the beanbag on your head, under your chin, between your knees, then throw it up in the air"
- Play the game Snakes and Ladders and model the use of positional/directional language, across, up, down, over, forwards, backwards.
- Support the children to construct the alphabet train floor puzzle and draw their attention to the position of different animals in relation to others, the cow is beside the bear, kangaroo in front of the lion, the pig is next to the ostrich etc.
- Set up a Small World Play Scene e.g. Garage, Doll's House, Farm, Happyland School bus or Ikea Playmat. Talk to the children about where to put certain objects. Take turns with child: they tell you where to put an object, then vice versa. Model positional language throughout the activity.
- Set up an obstacle course inside or outside, using boxes, cushions, chairs etc. Encourage use of positional and directional language (up, down, over, under or through, forwards or backwards). A fun twist on this could be where children take turns being "commentators"- commenting on what the participants are doing
- Model the use of lots of positional/directional language during sensory play (sand, water, gloop, shaving foam, play-dough)
- Use the traffic light poster to prepare children for quiet time/story time. Read the book "We're Going on a Bear Hunt". Describe the journey and how they go through, around, under, over each obstacle. Young children will love acting out the journey themselves around the room – using positional/directional language as directed. This book has great rhythm and can also be turned into a song.
- Read the book "Rosie's Walk". Talk to the children about the journey that Rosie takes and emphasise the positional/directional language, across, around, past, through. Ask the children to describe journeys that they have taken recently, e.g. going to preschool, the park, the shops, and encourage their use of positional/directional language. Encourage the children to make pictures based on the stories they have read and created.
- Take the children on a visit to the local library and look for books that focus on positional/directional language and read them to the children. This could be a nice event to get parents involved with.

Assessment: Does he/she begin to understand and recognise positional/directional language? Does he/she begin to use positional/directional language?

The Area Based Childhood Programme 2013-2016

The Atlantic Philanthropies

Early Learning Initiative
National College of Ireland

Docklands Early Numeracy Project

Indicative Evidence

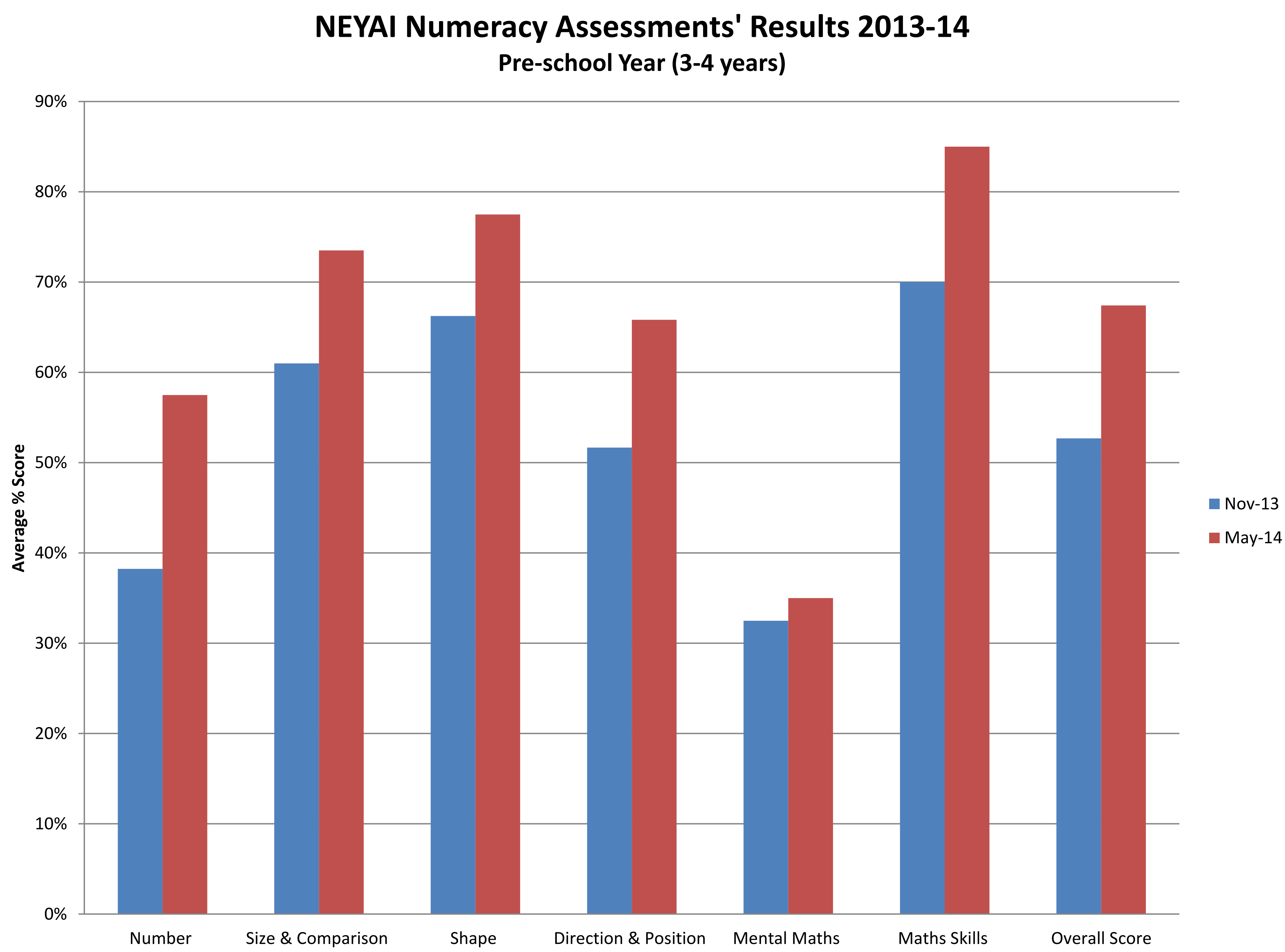
(Veerman and van Yperen 2007)

- ✓ 860 children (0-6 years) and their families take part each term
- ✓ Children's numeracy skills have improved and they are scoring to, if not above in some cases, national norms in Maths.

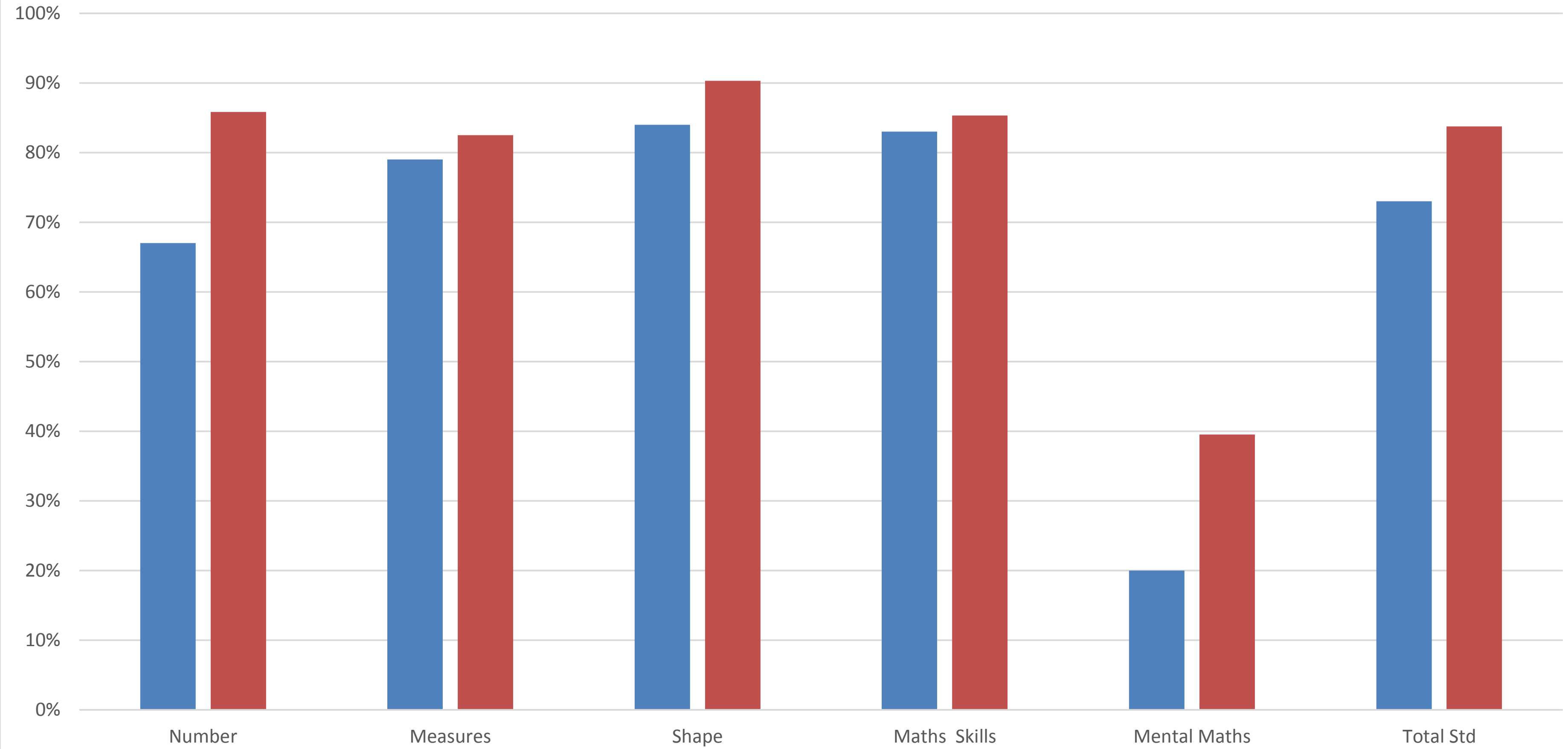
‘The children really grasped the concept, reinforced at home and in school. Maths was great fun.’

- ✓ Parents are more involved in their children’s learning (88% N=136)
- ✓ 97% (N=149) of parents would recommend the numeracy week/activities to a friend.

- ✓ ECCE practitioners are more skilled in supporting children's numeracy outcomes (99% N=457)
- ✓ Quality of their practice had improved (99% N=458)


NEYAI Numeracy Assessments' Results 2013-14
Pre-school Year (3-4 years)
Average % Score
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Number
Size & Comparison
Shape
Direction & Position
Mental Maths
Maths Skills
Overall Score
Nov-13
May-14

Standard Scores For NEYAI Vs Non NEYAI Centres



Summary	Number	Measures	Shape	Maths Skills	Mental Maths	Total Std
NEYAI Centres	67%	79%	84%	83%	20%	73%
Non NEYAI Centres	86%	82%	90%	85%	40%	84%
Delta	19%	3%	6%	2%	20%	11%

Impact on children (N=445)	Impact on parents (N=309)
Improved understanding of numeracy concepts (60% N=265) Enjoyed numeracy activities (31% N=140) Parents more involved (9% N=40)	Increased Involvement (47% N=146) Awareness & learning (28% N=88) Enjoyed activities (24% N=75)

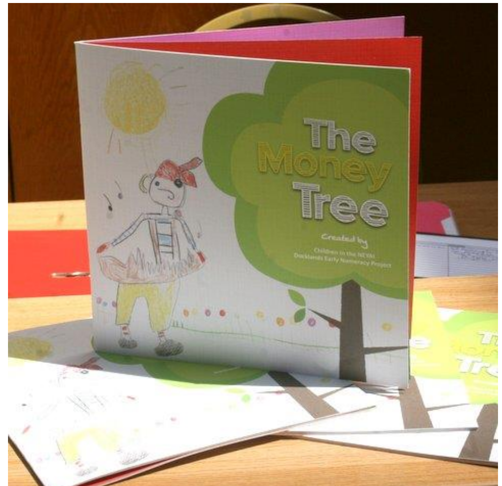
The children took a huge interest in measuring, weighing, they started seeing everything in the room as something to potentially measure or weigh. They also started using the terminology

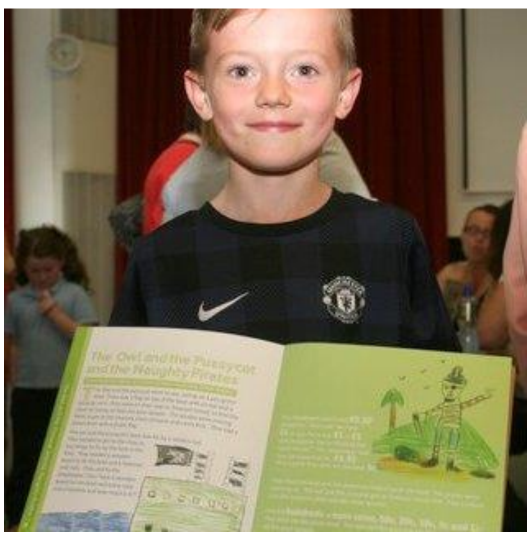
I felt the games we played in class and putting actions to rhymes worked very well. The children really grasped the idea of positional language as they could see it and understand rather than it being an abstract concept

The theme of money was easy for the children to relate to and they could experience it at first hand. We conducted the money value exercise for all children in the class. We took them to the stores and taught them how to use money

Children enjoyed the rhymes and they love hearing new counting rhymes. It has had a strong impact on the children because parents have said that the older children are currently trying to learn the counting rhymes, both at home and out and about.

Celebration Event





Implementing NEYAI Docklands
Early Numeracy Project
St Andrew's Child Care Service

Implementing NEYAI Docklands Early Numeracy Project
Holy Child Preschool

Conclusions/ Recommendations

- Process of **community action research** provides evidence of effective implementation as well as enabling continuous improvement
- **Aistear works** as a curriculum framework for planning, implementation & evaluation
- **Genuine community involvement** in decision process is crucial (through Working Group and Consortium Meetings).
- **Multiple methods for parents** to engage with (workshops, home based activity cards, Facebook and Curriculum Priority events in ECCE Services, Schools, through PHNs, Home Visitors, After Schools and Libraries)
- **Community wide focus** on numeracy fosters multi-sectoral working, involvement of parents, curriculum planning and better numeracy outcomes for children.

Future Challenges

- NEYAI Programme has ended so transferring into the Area Based Childhood Programme
- Consortium and working group larger
- New services coming on board
- Reviewing & improving the programme
- Involving parents
- Being prepared
- Time for communication (staff & parents)
- Handling on-going challenges: evaluations; changes in personnel & in the sector; bereavements; flooding; illness etc.

Early Learning Initiative

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Sharing Learning through the NEYAI Docklands Early Numeracy Programme Under Three's

Supporting parents, communities and schools in the education of children

Early Learning Initiative

National College of Ireland

Working in partnership with local communities to support educational journeys and achievements

- Address educational disadvantage and its impact on personal and career development
- Provide a range of innovative support programmes for children and their parents from early years to third level
- Uses community action research (Plan, Do, Review) to implement national policy and programmes

Collaborative Projects

NEYAI Early Numeracy Programme

Parent Child Home Programme

Language

Literacy

Numeracy

Educational & Career Guidance

Second & Third Level

Parenting Programmes

Primary

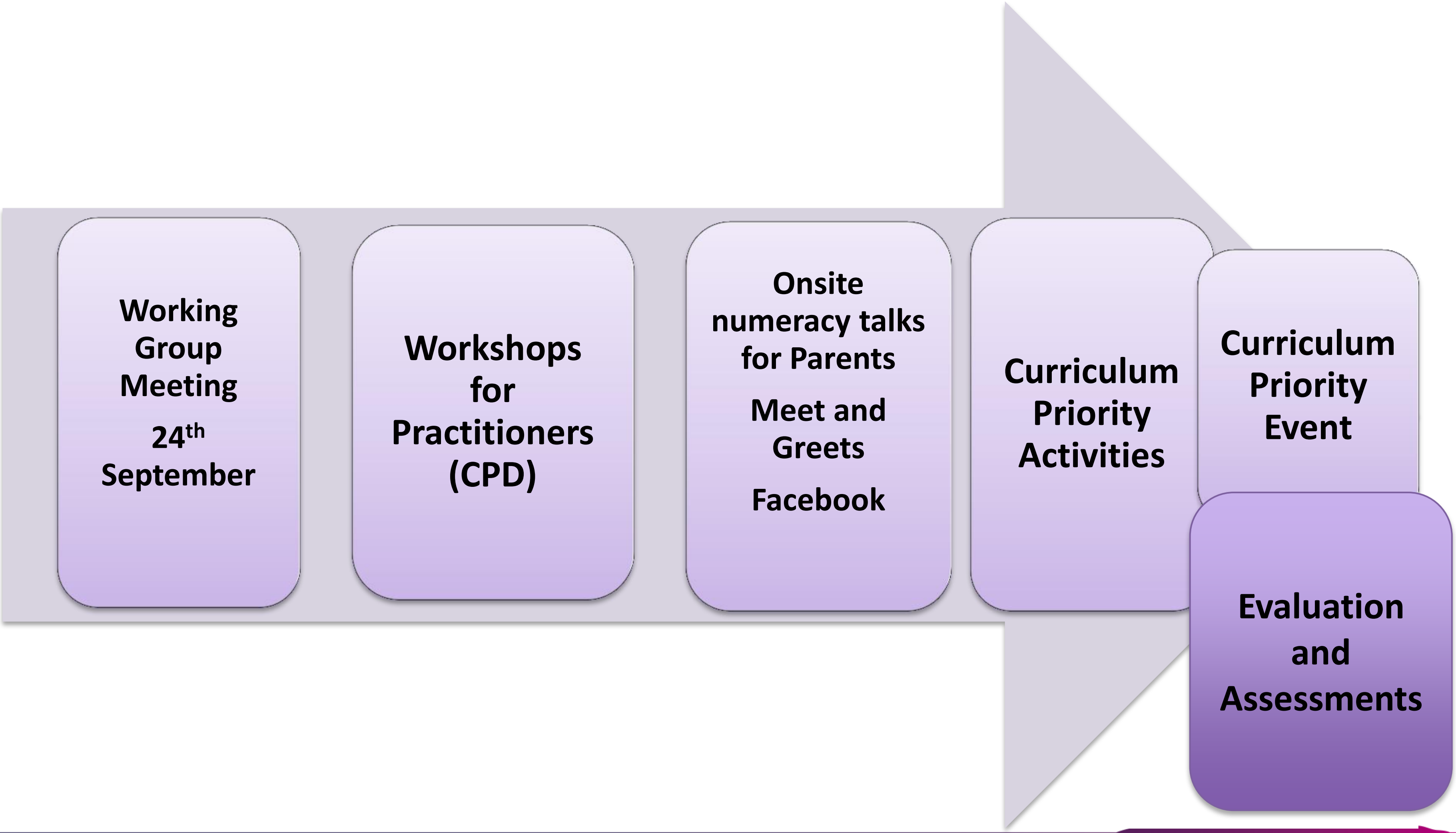
Rationale

- Internal ELI evaluations highlighted the low levels of numeracy in the Docklands area as well as the lack of support for parents in Mathematics (ELI 2010).
- National and international reports (DES 2005a; Surgenor et al 2006; Shiel et al 2007; Eivers et al 2010) emphasised how young people in Ireland were poorly prepared for future Mathematical needs as students and citizens.
- International literature review revealed
  - Opportunities for pre-schoolers to learn mathematics are often very inadequate (National Academy of Sciences 2009).
  - Enormous differences in the mathematical knowledge of children when they begin school (US Maths Recovery Council 2005; Northwestern University 2007; Every Child a Chance Trust 2009)
  - Those that are among the least advanced of their class remain so throughout their schooling and often give up on Mathematics.
- The lack of proficiency in maths-based subjects can be the trigger for non-completion at third level (HEA 2010).

Docklands Early Numeracy Project Objectives

- To improve the educational outcomes for children in the Docklands in numeracy
- To increase parental involvement in their children's development, learning and education by providing a variety of on-going supports for parents of young children.
- To support early childhood care and education workforce in implementing *Aistear, the Early Childhood Curriculum Framework* and *Síolta, The Quality Framework for Early Childhood Education*
- To ensure continuity and progression in Mathematical learning for children moving from home to early years settings to the local schools

Community Action Research Usual Schedule of Events



NEYAI Numeracy Curriculum Objectives

(Taken from *Aistear*)

	Babies (0-18 months)	Toddlers (12 months – 3 years)	Young Children (2½ -6 years)
Communication	Watches, listens and responds to adults when they use Mathematical language	Responds to and understands Mathematical language in everyday situations (p. 38)	• Develop basic counting skills (1-10) • Develop an understanding of the meaning and use of numbers in their environment • Understands and uses positional language such as up, down, out, behind
Exploring and Thinking	Experience and begin to understand simple cause and effect Develop the concept of object permanence	Compare, sort, categorise and order things Develop a sense of time, shape, size, space and place	• Classify, sequence, sort, match, look for and create patterns and shapes • Develop an understanding of concepts like measures (weight, height, volume, money, time) • Use mathematical symbols to give and record information, to describe and make sense of their own and others experience • Develop higher-order thinking skills such as problem-solving, predicting, analysing, questioning and justifying

Themes for 2014-2017

	Year 1 2014-2015	Year 2 2015-2016	Year 3 2016-2017
Term 1	Positional Language	Time	Money
Term 2	Counting	Measurement	Number
Term 3	Shapes	Sequence and Pattern	Symbols in the Environment

NEYAI Docklands Early Numeracy Working Group

- Chosen for your interest in the project and in early numeracy as well as your ability to lead the project in their setting.
- Meet 4 times a year approx. (June, September and December 2011)
- Communicating between services and working group (open, honest, critical, responsible)
- Responsible for developing, planning and implementing the programme at front-line service delivery level using the community action research process

Role of the Working Group

- Bring and share early numeracy expertise and experience from working with various age-groups
- Network with other settings
- Input re. Theme/ Focus of Curriculum Priority Week
- Input re. Resources, Activities etc.
- Communicate info. to their staff team
- Involve wider community

Co-ordinate curriculum priority week & related events in school/setting e.g. awareness of existing resources to complement c.p. wk activities

Docklands Early Numeracy Activity Week

Positional Language: on top, under, beside, in front of, next to, fit, inside, outside, between, around, on, off, into, out of, far, near, close to

Parents try this at home with your child……

Babies- (0-18months)

- Play "a where's teddy gone?" game and hide different objects as part of the game. Hide teddy under the blanket, behind your back, under your arm. Talk through what you are doing "I wonder where teddy is gone, is he under the blanket….?"

Observe- How does your child react? Smiles, laughs, follows you with their eyes, looks for teddy?

Wobblers/toddlers (12months-3 years)

- Play hide and seek with the child hiding first and you looking for them. Make sure to talk through what you are doing "Where's Lilly gone? Is she behind the door, Is she under the bed?…. Swap over and let your child find you.

Observe: How does your child react? Laughs, is having fun, copies you and uses positional language?

Young Children (2 ½ -6 years)

- Choose an item to hide (e.g. child's favourite toy) and give your child directions to find it (see positional language above). Swap over and get your child to hide the item and give you directions to find it.

Observe: How does your child react? Responds to your directions and find the item? Has difficulty following directions (in this case, make the directions simpler)?

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First card – Changes

A4 rather than A5

Different cards for different age groups

Emphasise conversations

Space for recording experiences & thoughts

Signed

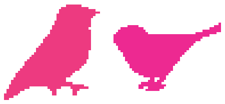
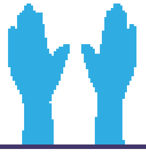
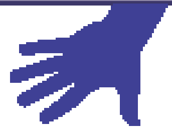
Docklands Early Numeracy Activity Week- Babies and Toddlers

Words to use: Up, down, front, back, in, out, on top, over, under, beside, behind, in front of, next to, fit, inside, outside, between, around, on, off, into, out of, far, near, close to, forward, backward, across, left and right.

Child's Name:

Positional / Directional Language

Using positional / directional language when singing, playing and talking with your child, will help them to learn these important early numeracy concepts.

	Monday	Tuesday	Wednesday	Thursday
				
Parent's Signature				
Play the games or sing the songs/ rhymes with your child and talk about it	**Two Little Dicky Birds** Two little dicky birds sitting On a wall (Use your finger to represent each bird) One named Peter, one named Paul. Fly Away Peter, fly Away Paul (Bring one finger at a time behind your back) Come back Peter, come back Paul! (Bring each finger back one at a time)	**Peek-a-boo** Play Peek-a-boo, hiding your face behind your hands and taking them away slowly. Talk through what you are doing, "Where's Mammy gone, Behind here..."? You can also encourage your child to hide their own face behind their hand and describe what they are doing, "Where is Jake?".	**Where's Teddy?** Hide teddy (or another familiar object), Under the blanket, Behind your back, Under your arm. Talk through what you are doing, "I wonder where teddy is gone, Is he Under the blanket..." Observe – How does your child react? Smiles, laughs, follows you with their eyes, looks for teddy?	**Finger Song** Up and down, round and round (draw circles in the air), Put your fingers on the ground. Over (hold hands above lap) Under, (below legs) In Between (between your legs) Now my fingers can't be seen! Hands In Front, hands Behind, now my hands I cannot find. Here's my left hand , here's my right, Hands and fingers back in sight (wriggle fingers).

Dear Parent: Do you think your child enjoyed this activity? Please circle one

☺ 😐 ☹

Friday	
Teacher can I have a sticker?	

The Area Based Childhood Programme 2013-2016

Early Learning Initiative
National College of Ireland

Docklands Early Numeracy Project

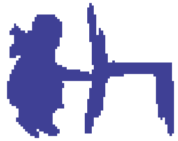
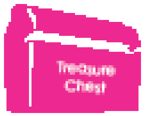
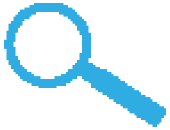
Docklands Early Numeracy Activity Week- Young Children- Pre-schoolers/Reception/Junior Infants

Words to use: Up, down, front, back, in, out, on top, over, under, beside, behind, in front of, next to, fit, inside, outside, between, around, on, off, into, out of, far, near, close to, forward, backward, across, left and right.

Child's Name:

Positional / Directional Language

Using positional / directional language when playing, singing and talking with your child, will help them to learn these important early numeracy concepts. Try these activities each day with your child and don't forget the rhymes at the back of the card.

	Monday	Tuesday	Wednesday	Thursday
				
Parent's Signature				
Play the games with your child, and talk about what you are both doing using the positional/ directional language	**Hide and Seek** Play Hide and Seek with your child hiding first and you looking for them. Talk through what you are doing, "where's Lily gone, is she behind the curtain, under the table, beside the couch? Swap over and let your child find you.	**Treasure Hunt** Choose an item to hide (e.g. child's favourite toy) and give your child hints to find it, using the positional language listed at top of card. Swap over and get your child to hide the item and give you directions to find it.	**I spy** Play the game 'I spy' and give your child clues about the item using positional language, e.g. It is Next To the mirror, Under the bed, Inside the shed. Swap over and let your child give you clues and practice their positional language.	**Obstacle Course** Set up a small obstacle course in the house, using cushions, chair, stool, table, blanket and anything else you can think of. Give your child directions, "Jump over the cushion, Go under the table, Go around the chair" If you are brave enough, swap over and let your child guide you!
Do you think your child enjoyed each activity? Please tick a smiley face.	😃 😐 ☹	😃 😐 ☹	😃 😐 ☹	😃 😐 ☹

Friday	
Teacher, can I please have a sticker?	

The Area Based Childhood Programme 2013-2016

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Early Learning Initiative
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Docklands Early Numeracy Project

Docklands Early Numeracy Activity Week

Shape and Space- Key Vocabulary: square, circle, triangle, rectangle, round, curved, straight, corner, flat, edges

ECCE Practitioners:

ECCE Practitioners- try an activity with the children every day this week, use rhymes & songs to support and extend the activities (overleaf).

Babies- (birth-18months)

- Sing shape rhymes to baby during care routines, tracing the shapes in the song on their palms, tummy etc.
- When carrying babies around the room/ setting, point out different shapes and say the name aloud. If possible, allow baby to touch these and enjoy their different textures and shapes.
- Allow children to experiment with, and understand how different shapes and sizes will fit into spaces e.g. stacking cups, nesting boxes or dolls, cardboard boxes of various sizes.
- Treasure basket- during quiet time, where you can be close and attentive, allow baby to explore the shape, form and texture of the objects in the treasure basket

Assessment- How does the child react? Responds with interest? Engages with the objects? Is willing to explore and experiment?

Wobblers/Toddlers (12months-3 years)

- Messy Play/ sensory play – adult models tracing of various lines (straight, curved lines, zig-zag lines) in shaving foam, wet and dry sand, gloop, dried pasta, rice, etc. Children have an opportunity to observe the adult closely and copy the designs if they wish. Adult then progresses to making shapes such as circles, saying the word circle as they make the curved line join.
- Play dough time- make some play dough with the children- this provides lots of rich opportunities for language development. Using cutters or using plastic knives, cut out shapes - adult models use of language such as straight line, curved etc.
- Printing Time- make a collection of similar shaped items e.g. bottle tops, jam-jar lids, plastic plates. Using paint, encourage children to make prints with these objects and discuss the size and shape of each print

Assessment: Does the child copy your actions? Make their own shapes? Is he/she beginning to notice simple shapes and patterns in pictures? Does he/she begin to talk about the shapes of everyday objects?

Young Children (2 ½ -6 years)

- Fun with shapes- make pictures using pre- cut shapes (paper, foam, card)- e.g. two circles, one large and one small could make a snowman, use squares and rectangles to make a robot. Display the pictures for all to see!
- Chalk Shapes- bring the children outside to an area where an adult has used chalk on the ground to draw large versions of lines and shapes (e.g. square, circle, triangle). Children can then walk, run, hop, skip around and into these shapes. When children are familiar with this activity, they can draw the lines and shapes themselves.
- The 'Feely Bag' game- adult places a number of plastic shapes into a cloth bag. Children take turns putting their hand into the bag, picking a shape and describing it aloud. When they have finished doing this they can guess the name of the shape. When introducing this game, spend plenty of time modelling the language and skills required to play.
- Construction play- play with blocks or open-ended construction materials (e.g. cardboard boxes) allows for many opportunities to explore shape and space. Try setting up a construction corner, providing a range of materials of various shapes, sizes, textures and weights. Try offering a mix of free-play and structured activities, spend plenty of time modelling the language and skills required to extend and enhance the learning experience.

Assessment: Does the child show an interest in shape and space by playing with shapes and making arrangements with shapes for example through engaging in construction activities or by talking about shapes or arrangements? Does he/she show awareness of similarities in shapes in the environment? Does the child show curiosity about and observation of shapes by talking about how they are the same and different

NEYAI is funded by Atlantic Philanthropies, Mount Street Club Trust, The Department of Education and Skills and the Department of Children and Youth Affairs

Docklands Early Numeracy Activity Week

Positional/Directional Language - Key Vocabulary: Up, down, front, back, in, out, on top, over, under, beside, behind, in front of, next to, fit, inside, outside, between, around, on, off, into, out of, far, near, close to, forward, backward, across, left and right.

ECCE Practitioners - try an activity with the children every day this week, use rhymes & songs (overleaf) and your own ideas to support and extend the activities.

Babies (birth-18months)

- Play Peek-a-boo, hiding your face behind your hands and taking them away slowly. Talk through what you are doing, "Where's Mary gone, Behind here..."? You can also encourage the children to hide their own face behind their hand and describe what they are doing, "Where is Jake?".
- Play a game of "where's Teddy gone?" and hide different objects as part of the game. Hide teddy under the blanket, behind your back, under your arm. Talk through what you are doing "I wonder where teddy is gone, is he under the blanket.....?"
- Give babies lots of opportunities to experience spatial awareness, letting them crawl or climb over cushions, under tables, in and out of boxes, through tunnels, inside tents and behind curtains, where possible describing to them what they are doing and giving them the language.
- Read the book "Where's Spot". Emphasise the positional/directional language e.g. behind the door, inside the clock, and give the children the chance to pull up the flaps. Use the toys in the baby room to make the story come to life.
- Read the book "Slow Snail". Emphasise the positional/directional language in the story e.g down her flower, over a stone. Trace your finger along the snail's path and encourage the children to do it also, while you describe what they are doing.
- Use lots of positional/directional language when babies are playing with bricks, blocks, stacking cups or stacking ring, describing what they or you are doing, "Look, we placed the yellow brick on top of the blue brick, the red cup inside the black cup..etc"
- Use lots of positional/directional language when babies are engaged in physical and outdoor play, "Look at you going up the slide, rolling under the table, climbing on top of the mat, crawling inside the tent, going behind the cupboard..etc"
- Emphasise positional/directional language during everyday activities such as changing and feeding time, "We'll get you up on top of the changing table, take your nappy off, find the cream inside the box..", "Let's all get our cups on the table"

Assessment: How does the child react? Engages with the games and activities? Does he/she begin to understand and recognise positional/directional language?

Wobblers/Toddlers (12 months-3 years)

- Emphasise positional/directional language when toddlers are engaged in physical and outdoor play, such as; "Look at you under the table, on top of the slide, under the tree..etc".
- Play Hide and Seek with the child hiding first and you looking for them. Make sure to talk through what you are doing "Where's Lily gone? Is she behind the door, is she under the table?"
- Play Treasure Hunt Game- while looking for an object hidden by the adult, model positional language e.g. "I wonder where teddy is gone, is he under the chair, let's look under here?" "Will we look up here?"
- Set up a Small World Play Scene e.g. Garage, Doll's House, Farm, Happyland School bus or Ikea Playmat. Talk to the children about where to put certain objects. Take turns with child: they tell you where to put an object, then vice versa.
- After a game/activity, use the traffic light poster to prepare toddlers for quiet time/story time.
- Read the book "Where's Spot". Emphasise the positional/directional language e.g. behind the door, inside the clock, and encourage the toddlers to pull up the flaps and talk about what they see. Use the toys you have in the room to make the story come to life.
- Read the book "Slow Snail". Emphasise the positional/directional language in the story e.g down her flower, over a stone. Encourage the toddlers to trace their fingers along the snail's path and talk about the story. Make a connection to outdoor play and the natural environment and collect props in the story such as flowers, stones and sticks. Watch out for snails in the garden or on trips to the park.
- Read the book "We're Going on Bear Hunt". Describe the journey and how they go through, around, under, over each obstacle. Toddlers will love acting out the journey themselves around the room – using positional/directional language as directed. This book has great rhythm and can also be turned into a song. Encourage the toddlers to make pictures based on the books they have read.
- Model and encourage the use of lots of positional/ directional language when playing with bricks, blocks, stacking cups or stacking ring, and pegboard, describing what they or you are doing, "Look, we placed the yellow brick on top of the blue brick, the red cup inside the black cup..etc"
- Model the use of lots of positional/directional language during sensory play (sand, water, gloop, shaving foam, play-dough)
- Take the children on a visit to the local library and look for books that focus on positional/directional language and read them to the children. This could be a nice event to get parents involved with.

Assessment: How does the child react? Does he/she begin to understand and recognise positional/directional language? Does he/she begin to use positional/directional language?"

Young Children (2 ½ -6 years)

- Play a game of "Simon Says" using an object like a beanbag. Each child gets a beanbag- adult gives directions as to where to place the beanbag- e.g. "Please put the beanbag on your head, under your chin, between your knees, then throw it up in the air"
- Play the game Snakes and Ladders and model the use of positional/directional language, across, up, down, over, forwards, backwards.
- Support the children to construct the alphabet train floor puzzle and draw their attention to the position of different animals in relation to others, the cow is beside the bear, kangaroo in front of the lion, the pig is next to the ostrich etc.
- Set up a Small World Play Scene e.g. Garage, Doll's House, Farm, Happyland School bus or Ikea Playmat. Talk to the children about where to put certain objects. Take turns with child: they tell you where to put an object, then vice versa. Model positional language throughout the activity.
- Set up an obstacle course inside or outside, using boxes, cushions, chairs etc. Encourage use of positional and directional language (up, down, over, under or through, forwards or backwards). A fun twist on this could be where children take turns being "commentators"- commenting on what the participants are doing
- Model the use of lots of positional/directional language during sensory play (sand, water, gloop, shaving foam, play-dough)
- Use the traffic light poster to prepare children for quiet time/story time. Read the book "We're Going on a Bear Hunt". Describe the journey and how they go through, around, under, over each obstacle. Young children will love acting out the journey themselves around the room – using positional/directional language as directed. This book has great rhythm and can also be turned into a song.
- Read the book "Rosie's Walk". Talk to the children about the journey that Rosie takes and emphasise the positional/ directional language, across, around, past, through. Ask the children to describe journeys that they have taken recently, e.g. going to preschool, the park, the shops, and encourage their use of positional/directional language. Encourage the children to make pictures based on the stories they have read and created.
- Take the children on a visit to the local library and look for books that focus on positional/ directional language and read them to the children. This could be a nice event to get parents involved with.

Assessment: Does he/she begin to understand and recognise positional/directional language? Does he/she begin to use positional/directional language?

The Area Based Childhood Programme 2013-2016

The Atlantic Philanthropies

Early Learning Initiative
National College of Ireland

Docklands Early Numeracy Project

Indicative Evidence

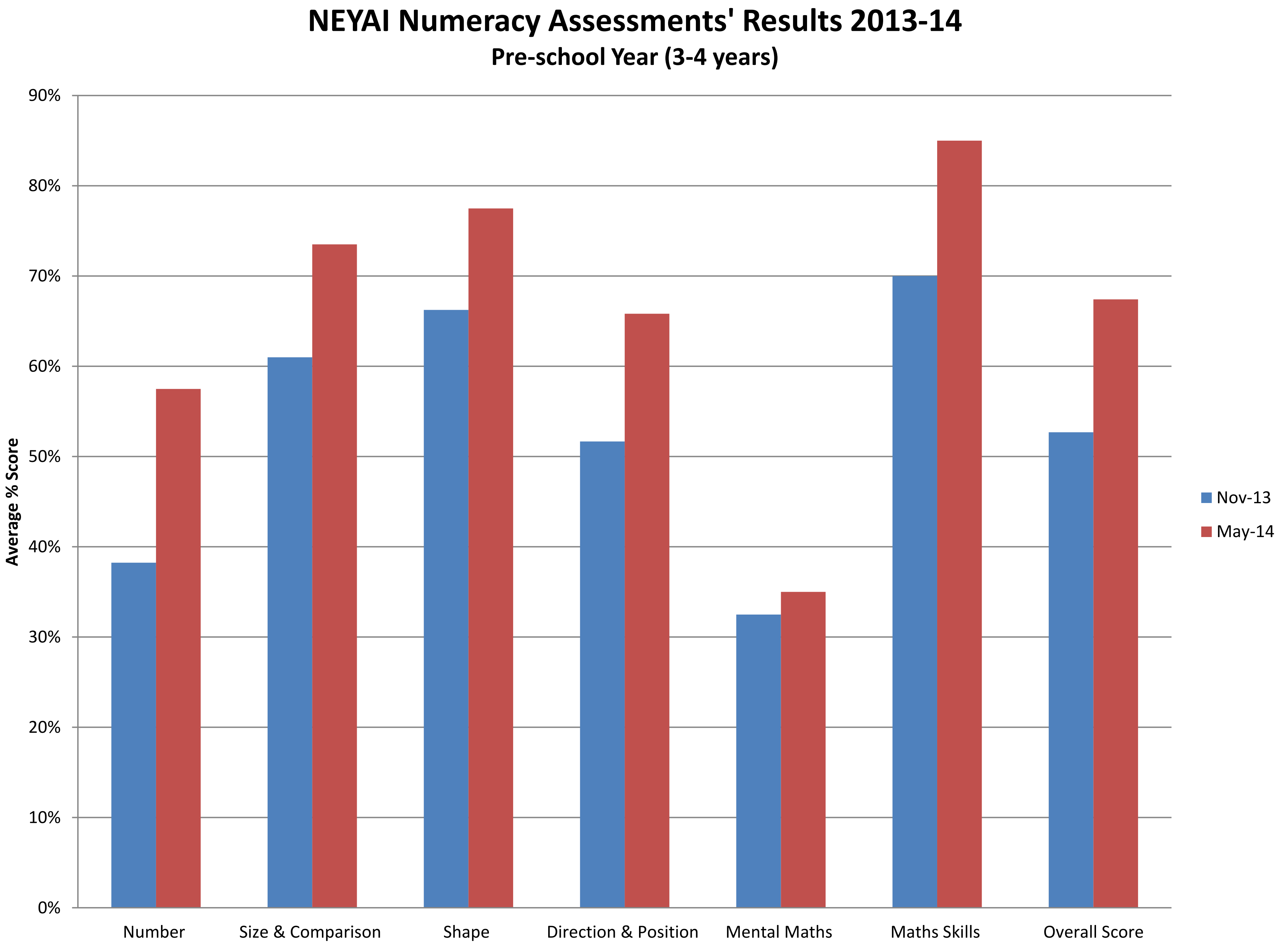
(Veerman and van Yperen 2007)

- ✓ 860 children (0-6 years) and their families take part each term
- ✓ Children's numeracy skills have improved and they are scoring to, if not above in some cases, national norms in Maths.

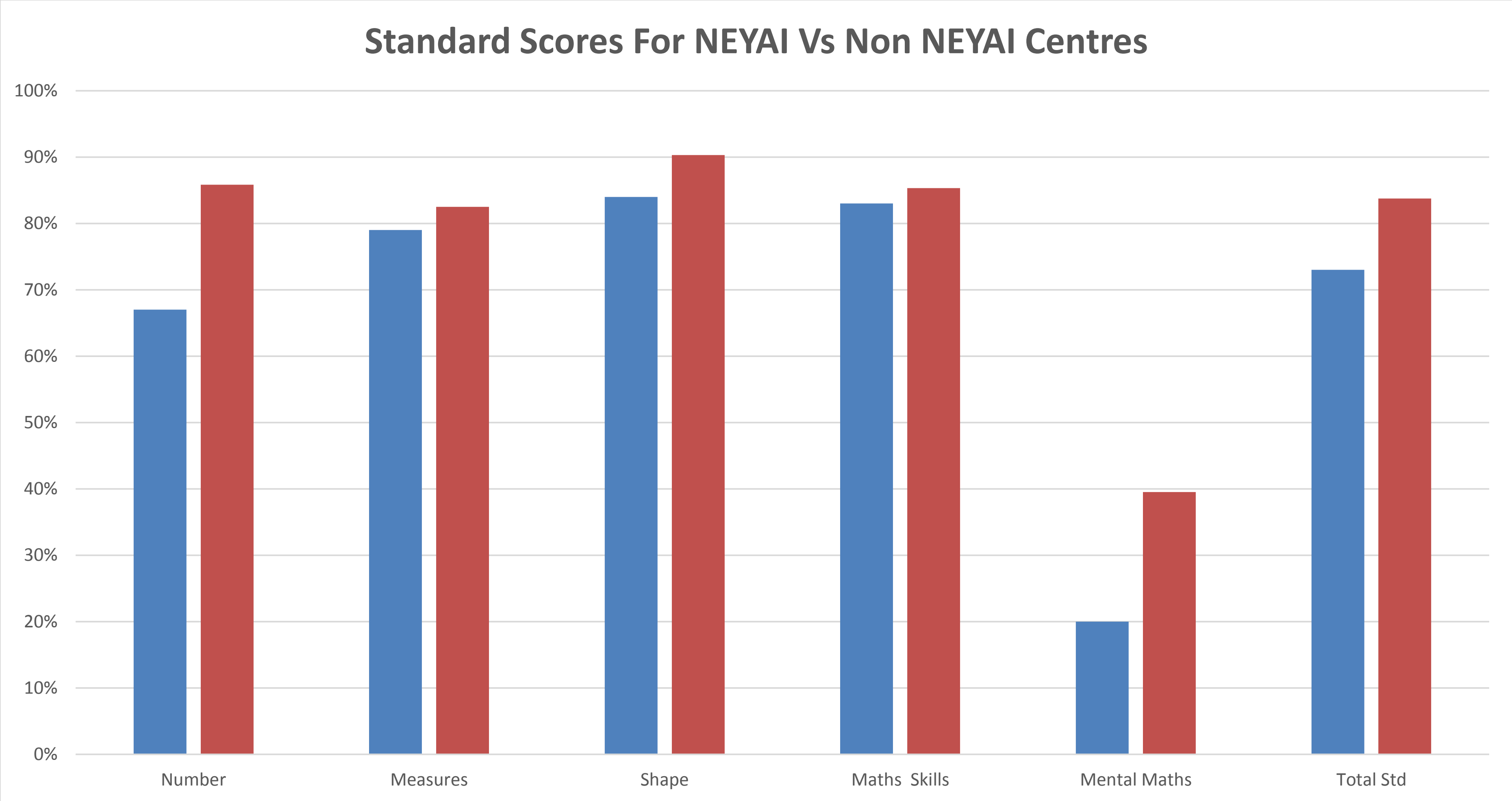
'The children really grasped the concept, reinforced at home and in school. Maths was great fun.'

- ✓ Parents are more involved in their children's learning (88% N=136)
- ✓ 97% (N=149) of parents would recommend the numeracy week/activities to a friend.

- ✓ ECCE practitioners are more skilled in supporting children's numeracy outcomes (99% N=457)
- ✓ Quality of their practice had improved (99% N=458)



NEYAI Numeracy Assessments' Results 2013-14
Pre-school Year (3-4 years)
Average % Score
90%
80%
70%
60%
50%
40%
30%
20%
10%
0%
Number
Size & Comparison
Shape
Direction & Position
Mental Maths
Maths Skills
Overall Score
Nov-13
May-14



Summary	Number	Measures	Shape	Maths Skills	Mental Maths	Total Std
NEYAI Centres	67%	79%	84%	83%	20%	73%
Non NEYAI Centres	86%	82%	90%	85%	40%	84%
Delta	19%	3%	6%	2%	20%	11%

Impact on children (N=445)	Impact on parents (N=309)
Improved understanding of numeracy concepts (60% N=265)	Increased Involvement (47% N=146)
Enjoyed numeracy activities (31% N=140)	Awareness & learning (28% N=88)
Parents more involved (9% N=40)	Enjoyed activities (24% N=75)

The children took a huge interest in measuring, weighing, they started seeing everything in the room as something to potentially measure or weigh. They also started using the terminology

I felt the games we played in class and putting actions to rhymes worked very well. The children really grasped the idea of positional language as they could see it and understand rather than it being an abstract concept

The theme of money was easy for the children to relate to and they could experience it at first hand. We conducted the money value exercise for all children in the class. We took them to the stores and taught them how to use money

Children enjoyed the rhymes and they love hearing new counting rhymes. It has had a strong impact on the children because parents have said that the older children are currently trying to learn the counting rhymes, both at home and out and about.

Implementing NEYAI Docklands Early Numeracy Project St Andrew's Child Care Service

Aistear Guidelines

Supporting learning and development through assessment

PCHP Assessment Cycle

1. Assessment

Observations
Conversations
Tasks
CBTs & PACTs

2. Identifying Learning Needs

Children & Parents

3. Planning for Learning

Language
Approaches
Activities
Resources

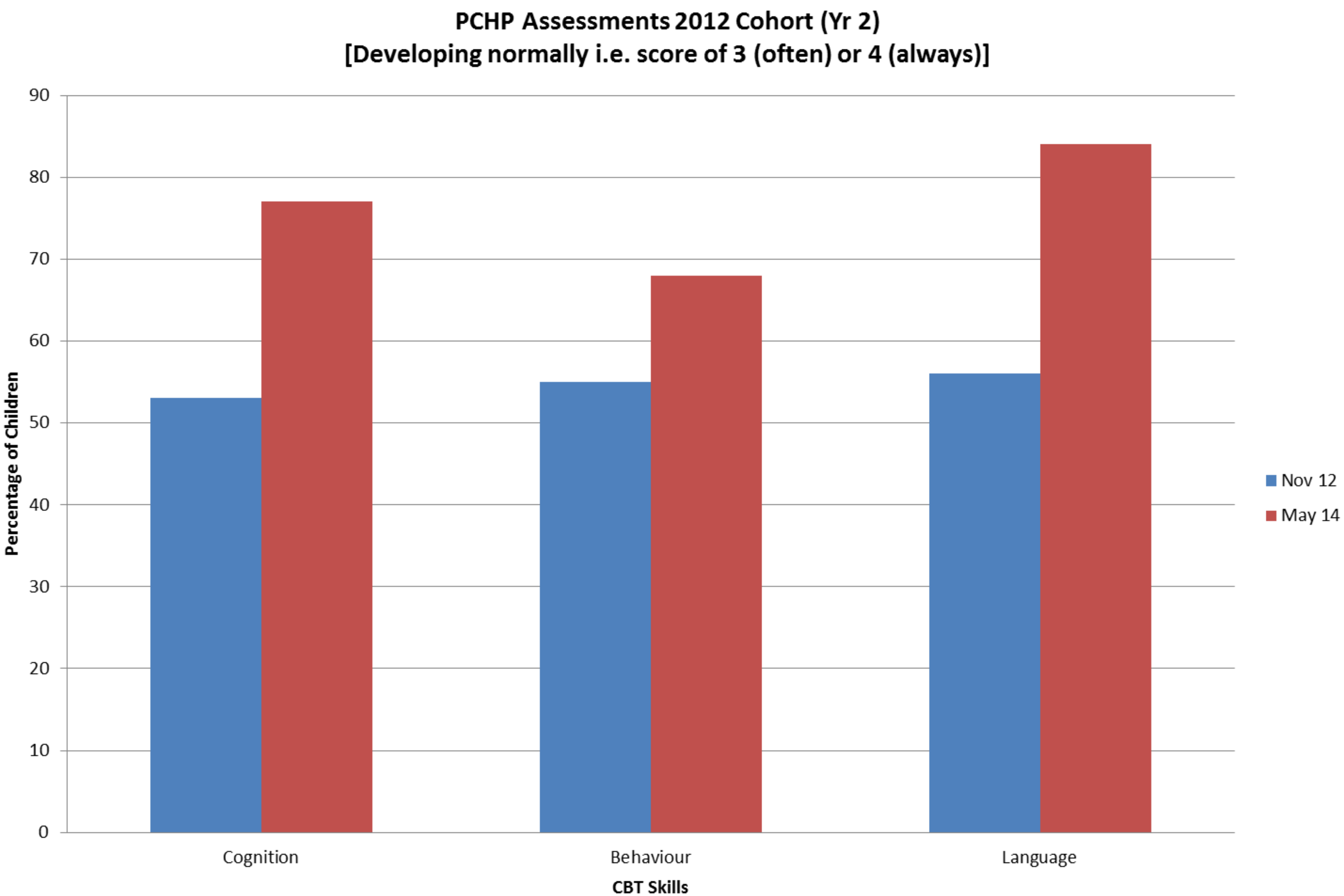
4. Modeling & Learning

Talking
Reading
Playing

Síolta

S7: Curriculum
S8:Planning and Evaluation

Implementing NEYAI Docklands Early Numeracy Project
Parent Child Home Programme


PCHP Assessments 2012 Cohort (Yr 2)
[Developing normally i.e. score of 3 (often) or 4 (always)]
Percentage of Children
90
80
70
60
50
40
30
20
10
0
Cognition
Behaviour
Language
CBT Skills
Nov 12
May 14

Conclusions/ Recommendations

- Process of **community action research** provides evidence of effective implementation as well as enabling continuous improvement
- **Aistear works** as a curriculum framework for planning, implementation & evaluation
- **Genuine community involvement** in decision process is crucial (through Working Group and Consortium Meetings).
- **Multiple methods for parents** to engage with (workshops, home based activity cards, Facebook and Curriculum Priority events in ECCE Services, Schools, through PHNs, Home Visitors, After Schools and Libraries)
- **Community wide focus** on numeracy fosters multi-sectoral working, involvement of parents, curriculum planning and better numeracy outcomes for children.

Future Challenges

- NEYAI Programme has ended so transferring into the Area Based Childhood Programme
- Consortium and working group larger
- New services coming on board
- Reviewing & improving the programme
- Involving parents
- Being prepared
- Time for communication (Staff & parents)
- Handling on-going challenges: evaluations; changes in personnel & in the sector; bereavements; flooding; illness etc.

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Coláiste Phádraig
St Patrick's College
Droim Conrach | Drumcondra

A College of
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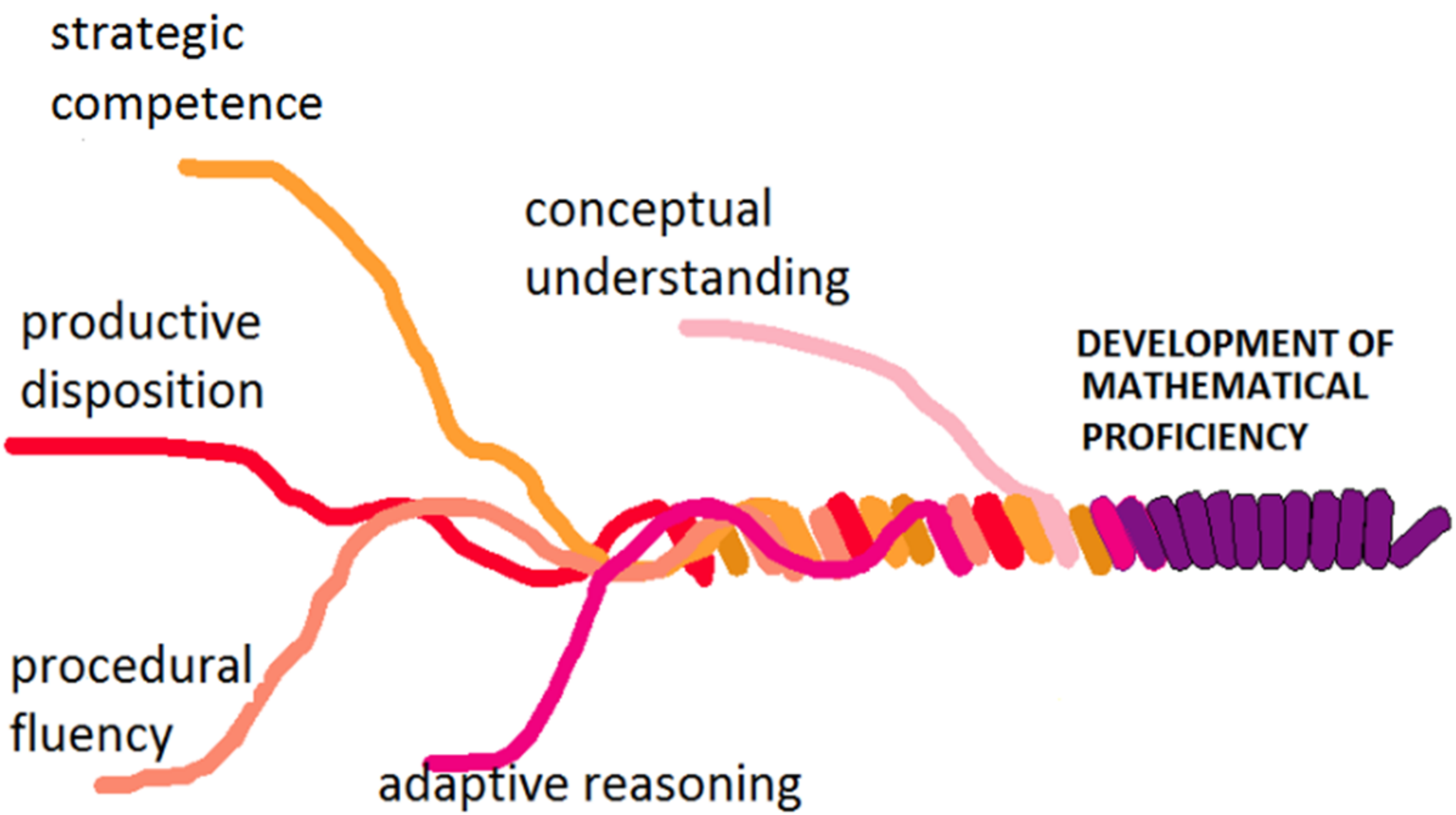
Young children communicating their mathematical thinking and understanding

Ross Ó Corráin and Liz Dunphy

www.spd.dcu.ie

Key Aim: Mathematical Proficiency



Adaptive Reasoning

Capacity for logical thought, reflection, explanation, and justification.

… the justification of one's work. This justification can be both formal and informal. Individuals clarify their reasoning by talking about concepts and procedures and giving good reasons for the strategies that they are employing. (US National Research Council, 2001, pp. 116-113)

Standards for Early Childhood Mathematics Education

Recommendation 14

As important as mathematical content are general mathematical processes such as problem solving , reasoning and proof, communication, connections, and representation; specific mathematical processes such as organising information, patterning and composing ; and habits of mind such as curiosity, imagination, inventiveness, persistence, willingness to experiment, and sensitivity to patterns. All should be involved in a high-quality mathematics program.

(Clements, Sarama and DiBiase 2004, p. 3)

Mathematizing is Key

…children interpreting and expressing their everyday experiences in mathematical form and understanding the relations between the two (Ginsburg, 2009)

…generalising concepts and situations first understood on an intuitive and informal level in the context of every day activity into mathematical terms… (National Research Council, 2009)

Key Implication from Research Report

Connecting, communicating, reasoning

argumentation, justifying, representing, problem solving and generalising should permeate all learning and teaching activities …

These processes are implicated in mathematization.

(Research Report No. 17, Chapter 1)

Towards A Change in Emphasis

Towards a revision of mathematics learning and teaching with the youngest children?

A change in pedagogy towards one that emphasises mathematization.

Researching Mathematatization

How can the teacher support young children's ability to mathematize?

- **Research participants:** My class
- **Methodology**: Action Research
- **Instruments**: Video, observation notes, video journal, children's mathematical representations

Strategies

Supporting mathematatization through…

- Children's mathematical representations (drawing, mark-making, writing, *photographs*)
- Engaging contexts for mathematical tasks (helping teacher, Talking Tom application)

Egg box Task

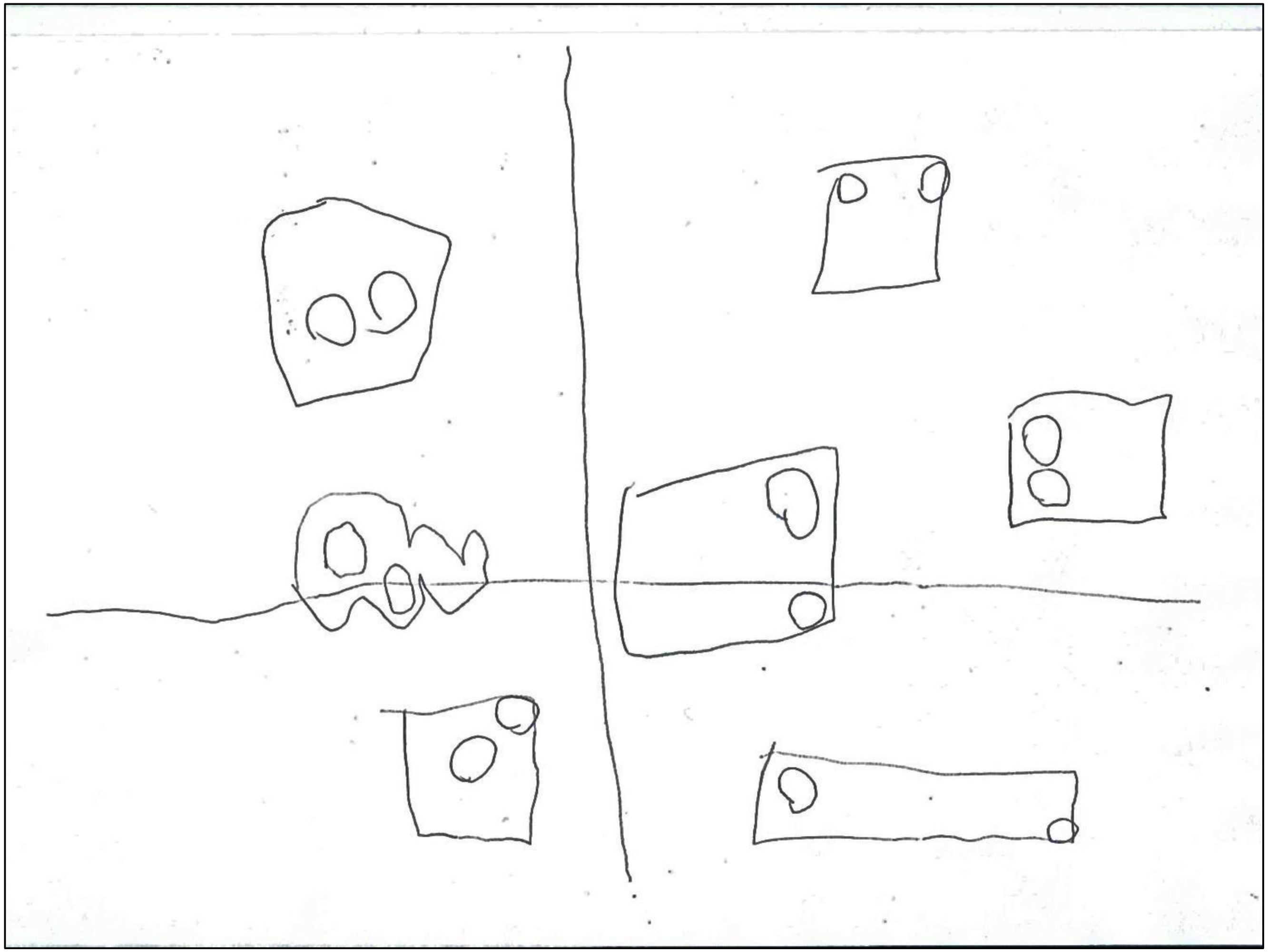
Strategy: Supporting mathematization through mathematical representations

- Children given egg box (6 spaces) and 2 eggs
- Find as many ways as possible the 2 eggs can be arranged in box
- Keep track of answers with paper and pencil

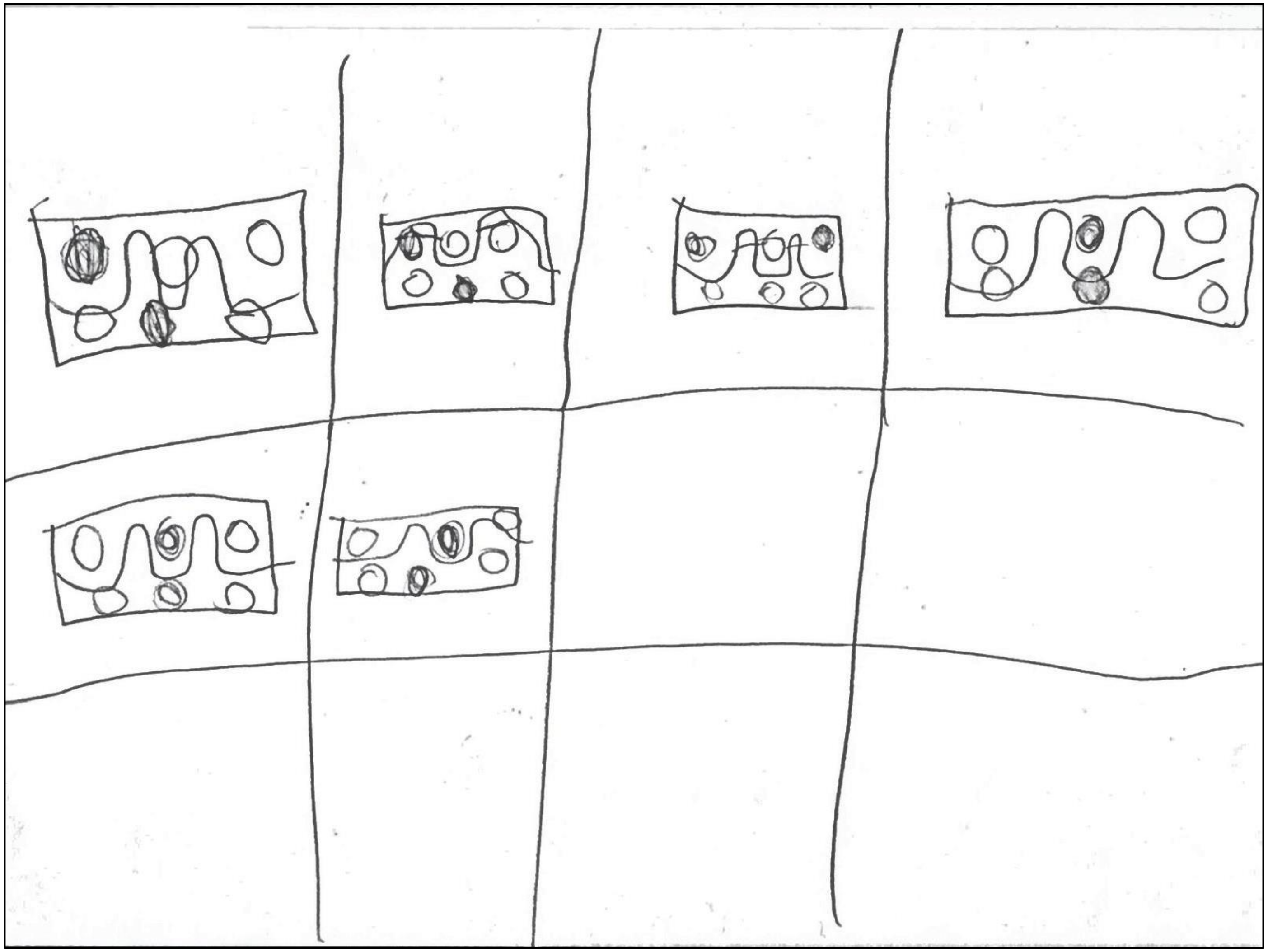
Egg box Transcript

- Read in pairs
- *What role does teacher play in supporting mathematizing?*

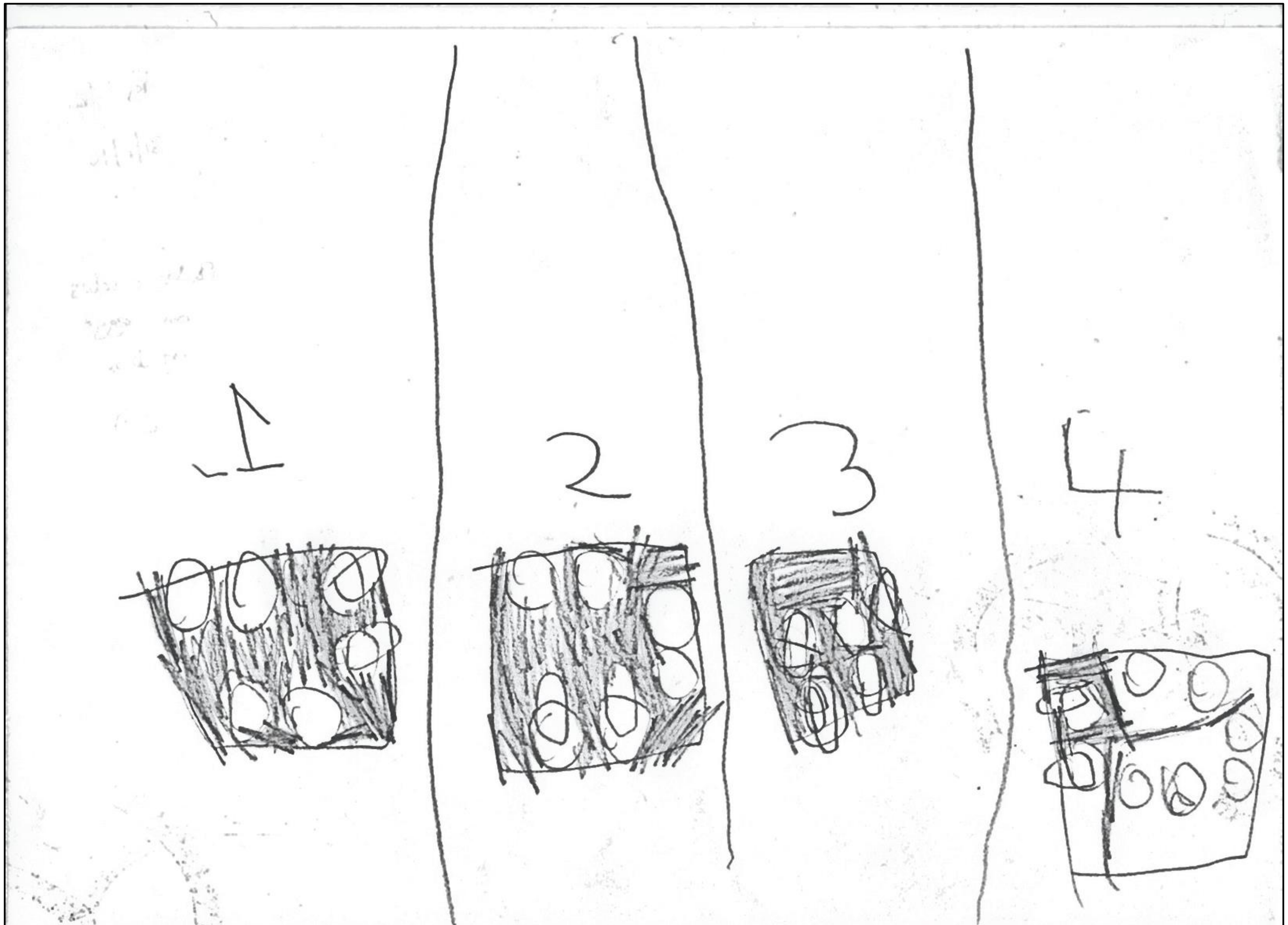
Katie



Lizzy



Fergie



Tower Task

Strategies: Supporting mathematization through meaningful mathematical context and digital photography

- Children set measurement task by Talking Tom

Tower Task

Video: *How does digital photography support mathematization in this clip?*

In Summary

Successful mathematization came through a combination of …

- **A meaningful context** for mathematical tasks : Talking Tom
- The **role of teacher**
- **Digital photographs**
- **Small group work**

References

Task Source: Cook, G., Jones, L., Murphy, C. & Thumpston, G. (1997). *Enriching early mathematical learning*. Buckingham: Open University Press

Digital Resource: Talking Tom application www.talkingtom.com

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Coláiste Phádraig
St Patrick's College
Droim Conrach | Drumcondra

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Dublin City University

Supporting children at risk of experiencing difficulties in early mathematics

Joe Travers and Órla Mc Kiernan

www.spd.dcu.ie

Early intervention and differentiation in mathematics

- Understand the rationale and importance of early intervention
- Examine two conceptual frameworks for early intervention
- Examine some evidence based approaches to intervention

Rationale and importance

- The research evidence from Griffin et al (1994); Hughes(1986); Gelman and Gallistel (1978); Mulligan (2011).
- The critical importance of the development of counting skills
- Inextricable link between the development of counting and number skills
- Influence of Piaget and the place of counting skills in the early mathematics curriculum

Conceptual framework one: Principles of counting

- Gelman and Gallistel outline five principles of counting
- Extremely useful framework for assessing, observing, analysing, and teaching early counting skills to pupils with SEN or at risk of experiencing difficulties in mathematics

Early intervention
Five principles of counting

- The one-one principle
- The stable order principle
- The cardinal principle
- The abstraction principle
- The order-irrelevance principle

Five principles of counting

- The meaning of the principle
- Common errors children make
- Appropriate activities to develop the principle

The one-one principle

- Matching of counting words to the items to be counted
- Need to recite the counting words in order
- Touch and count each item only once
- Co-ordinate touch and count so they occur at the same time

Common errors with the one-one principle

- touching an item more than once
- missing an item altogether
- repeating the counting name
- missing a counting name
- lack of co-ordination between the touch and the count
- counting beyond the number in the set

Appropriate activities to develop the principle

- reciting the number words in order- songs, rhymes, stories, rhythmic counting
- the child has to co-ordinate verbal, visual and motor components- they need support to develop keeping track strategies
- moving or marking items when counted
- counting the same set in different ways
- counting sounds, movements, items (touchable, moveable, and not)

The stable -order principle

- The counting words must be used in a repeatable stable order and be as long as the number of items to be counted
- No recognisable pattern up to thirteen
- Children often apply a non conventional sequence consistently
- Songs, rhymes, stories and rhythmic counting

The cardinal principle

- The final number in a set represents how many are in the set
- Dependent on the previous principles
- Child needs to use a counting word for each object, count each object only once, stop at the correct place, use the counting words in the right order and know that the last number represents how many are in the set

The cardinal principle

- link between counting and cardinality is crucial as it involves understanding that counting has a purpose or end product
- Different levels of development: response desired by adults to “how many?”
- Often they give the full count as a response
- Shift from the counting meaning of the last number to the cardinal meaning of the number

The cardinal principle

- Essential for learning to count on where the first number is given a cardinal meaning and the second a count meaning
- Activities: many experiences of counting with a purpose- how many?
- Number line with a link between numeral and its cardinal number
- Subitising the number of objects in a set
- Use of finger patterns

The abstraction principle

- The previous three how-to-count procedures can be applied to any counting situation real or imagined
- Previously thought that children should only count items that were identical
- Fed the idea that pre-number activities should be based on sorting and classifying
- Development facilitated by the practical experience of counting any set compiled

The abstraction principle

- Activities: counting sets of unlike objects as well as like…four things V four cars
- Part of transition to understanding that the adjective "four" describes the set not the cars unlike the adjective blue or small
- Counting items, sounds, movements
- Counting items that can be moved and not moved
- Counting items that are hidden

The order-irrelevance principle

- You can apply the how-to-count principles to a set of objects in any order
- When this is established children know that:
- Something counted is a thing, not a one or a two
- The counting numbers are used as counting tags for the objects to be counted and once the count is over they no longer belong to those objects

The order-irrelevance principle

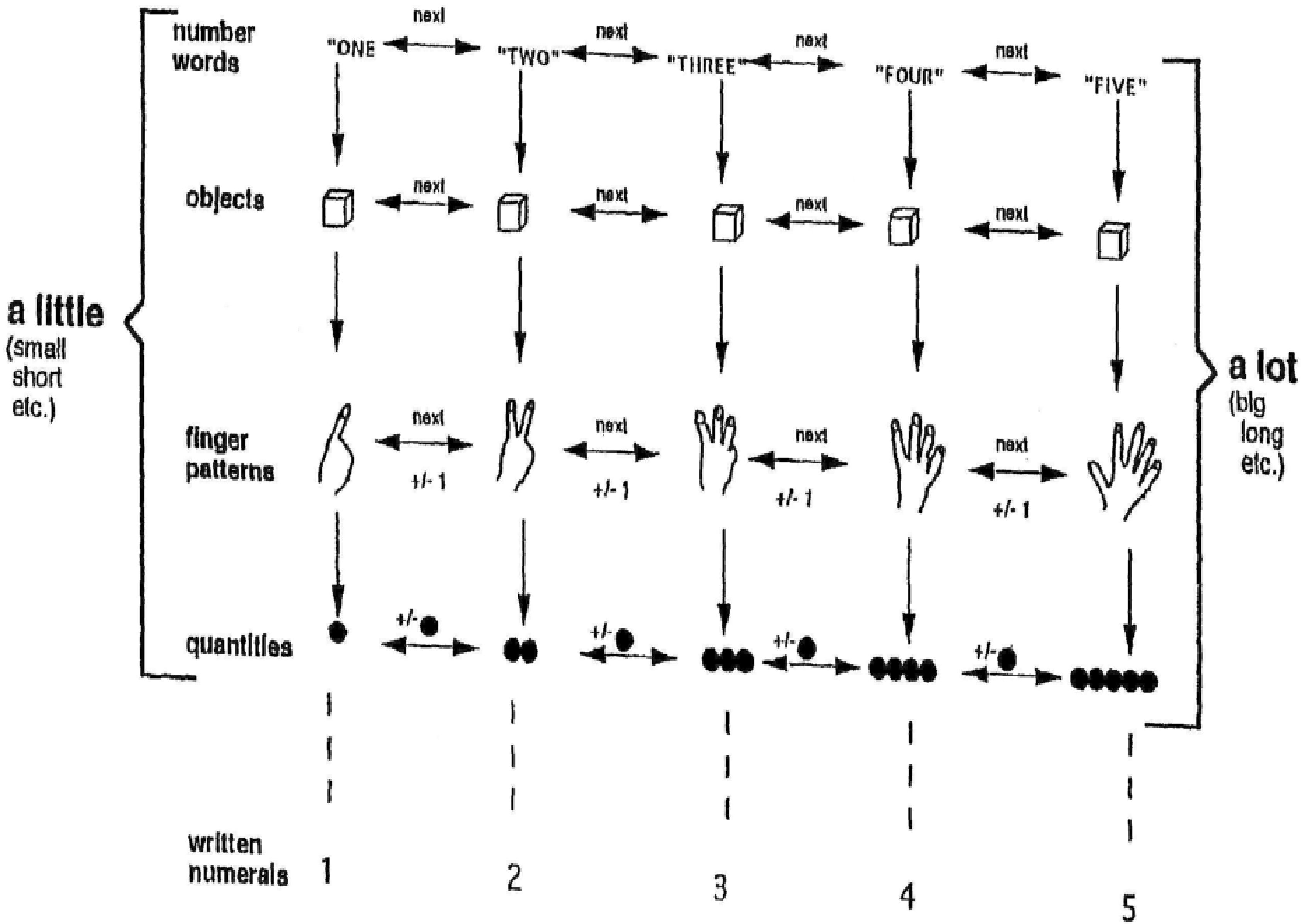
- Doesn't matter which number word is assigned to which object, it does not affect the cardinal number of the set
- The understanding of this principle comes much later. Children learn how to count before they fully understand the implications of their actions (Baroody, 1987)

The order -irrelevance principle

- Activities: counting left to right and right to left with the same row of objects
- Counting the same set but starting with a different "one" each time

Conceptual framework two: Central conceptual structure of number

- The psychology of early number development: Mental Counting Line- (Griffin et al., 1994)
- It describes the knowledge that appears to underlie successful learning of arithmetic
- "Central" as it forms a core on which all subsequent learning is based and their absence constitutes the main barrier to learning
- Number words: the child can recognise and generate the number words
- Objects: the child can count using the one-one and stable order principles


number words
"ONE
next
"TWO"
next
"THREE"
next
"FOUR"
next
"FIVE"
objects
next
next
next
next
a little
(small
short
etc.)
a lot
(big
long
etc.)
finger patterns
next
+/- 1
next
+/- 1
next
+/- 1
next
+/- 1
quantities
+/-
+/-
+/-
+/-
written numerals
1
2
3
4
5

Mental counting line- explained

- Finger patterns: the child understands that each number label has a set size associated with it – cardinal understanding
- Quantities: the child understands that movement from one of these set sizes to the next involves the addition or substraction of one unit
- Written numerals: the child can recognise written numerals and how they are linked to the set sizes (Match, select and name technique)

Number Knowledge Test

1. Let's see if you count from 1 to 10. Go ahead.
2. Can you count these counters for me? (Place 3 counters in front of child)
3. Show a group of 2 counters beside a group of 5 counters and ask: Which pile has more?
4. Show a group of 8 counters beside a group of 3 counters and ask: Which pile has less?
5. Show a line of yellow and red counters and ask child to count just the yellow counters.

Mental counting line games sranumberworlds.com

- Number Worlds programme and research evidence from the Irish context (Mullan and Travers, 2010)
- Teaching pupils the increment rule
- Teaching pupils to respond to questions about relative magnitude in the absence of any concrete sets of objects
- See also the counting section of Pitt (2001) Ready, Set, Go- Maths

Conceptual framework three: AMPS

Coláiste Phádraig
St Patrick's College
Droim Conrach | Drumcondra

- Awareness of mathematical pattern and structure (AMPS) (Mulligan, 2011)

- Children's acquisition of pattern and structure may be fundamental to establishing the root causes of mathematical learning difficulties (Mulligan, 2011)
- Across primary school children with low numerical achievement elicit descriptive and idiosyncratic images; they focus on non-mathematical aspects and surface characteristics of visual cues. They produce poorly organised, pictorial and iconic representations lacking in structure. They lack visualisation skills and flexibility in thinking.

AMPS

- AMPS is a construct that may help teachers recognise early difficulties in mathematics learning and intervene
- Lack of AMPS can impede later development of multiplication, division, measurement concepts, fractions and proportional reasoning

Stages of development of AMPS

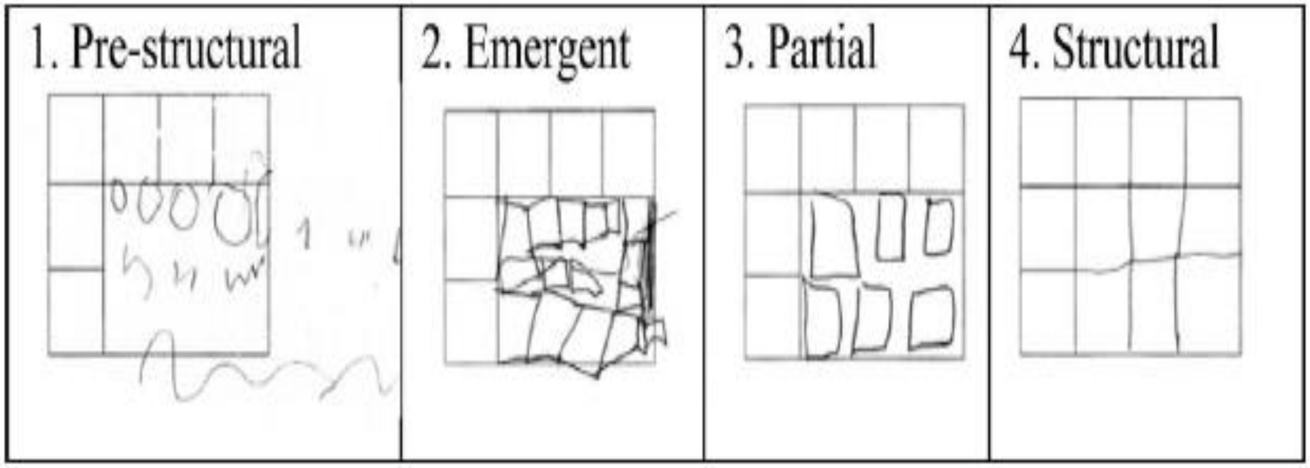


Figure 1. Typical responses by four different 6-year-old students to an area task, by stages of structural development.

Pedagogical intervention

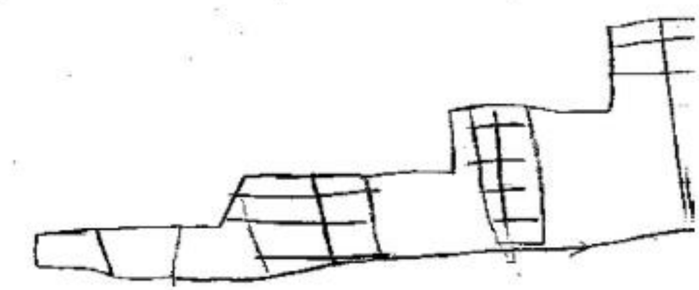


Figure 7. Staircase pattern by twos: drawing by copying model.

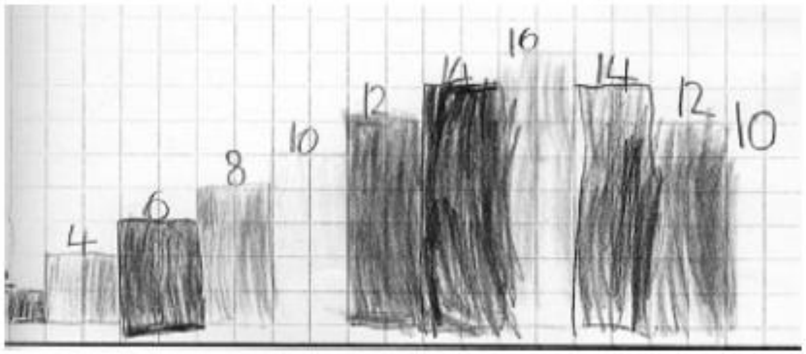


Figure 8. Staircase pattern by twos: post-intervention drawing of model from memory.

Pedagogical intervention

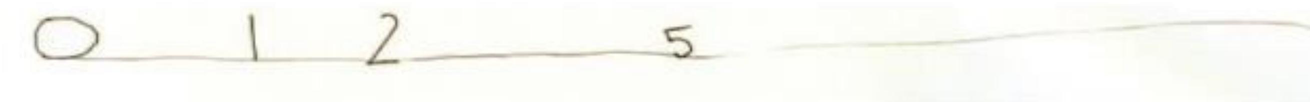


Figure 9. Student's initial attempt to estimate and record numerals to 10 on the number line (emergent structural stage).

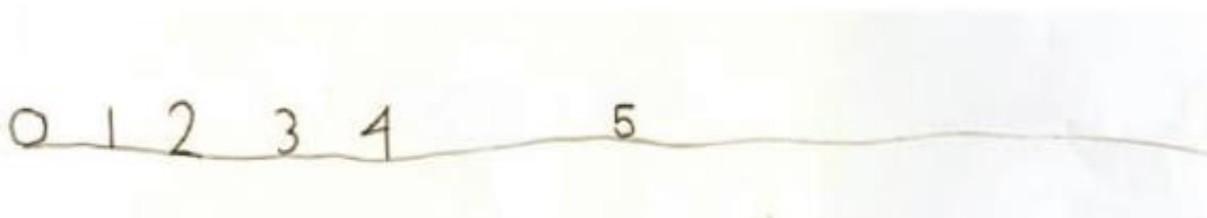


Figure 10. Student's second attempt to estimate and record numerals to 10 on the number line (partial structural stage).

Figure 11. Student's third attempt to estimate and record numerals to 10 on the number line (structural stage).

Pedagogical intervention

- Task: to assess understanding of the structure of the counting sequence- reproduce from memory a number line
- Child 6 years, 4 months
- *"Oh I didn't know the numbers are the same space between, even when you get a fat number, so 2 is 1 space bigger than 1, and 3 is one bigger than 2. So I have to make the spaces the same size going along the line even if my numbers are getting bigger and you get numbers like 99..so it doesn't matter if I count a long way, past 100, the space is the same long cause its one more every time…Oh I get it now"*

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Background Paper and Brief for the development of a new Primary Mathematics Curriculum

NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

October 2016

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Background paper for the development of a new primary mathematics curriculum

The current Primary School Mathematics Curriculum (PSMC) (Department of Education and Science [DES], 1999) was introduced in 1999, with in-service for mathematics provided in 2001–02, and implementation beginning in 2002–03 (DES, 2005). Much has changed and happened since then. In meeting the demands of unprecedented societal and educational change, it is important to review and update the curriculum to ensure children are afforded a high-quality, coherent, and more relevant mathematics education that will contribute towards their personal and academic learning and development. This background paper is not exhaustive but will, it is hoped, provoke rich discussion and provide emergent signposts towards the development of a new Primary Mathematics Curriculum (PMC[1]). The paper begins by setting out the context for change and posing the question, *What is mathematics?* before offering a brief synopsis and critique of the PSMC.

This background paper draws on an extensive suite of evidence which includes relevant national and international data and research. In particular, it utilises the National Council for Curriculum and Assessment's (NCCA) curriculum reviews (2005, 2008) and evaluations by the Department of Education and Skills (2005, 2010), the two recent mathematics research reports (NCCA Reports 17 and 18, 2014), and the international audit of mathematics curricula (Burke, 2014) commissioned by the NCCA. Findings from focus groups carried out to elicit teachers' and principals' views, beliefs and values regarding mathematics learning and pedagogy, and their ideas regarding the development of a new mathematics curriculum, are also included.

[1] For the purposes of distinguishing between the current and new primary math curricula, the 1999 curriculum will be abbreviated as PSMC whereas the new primary curriculum will be abbreviated as PMC.

The following online research reports, summary and audit are recommended in support of this background paper:

- Dunphy, E., Dooley, T., and Shiel, G. (2014). *Mathematics in Early Childhood and Primary Education*. Research Report 17, National Council for Curriculum and Assessment, Dublin. Available at http://ncca.ie/en/Publications/Reports/NCCA_Research_Report_17.pdf

- Dooley, T., Dunphy, E., and Shiel, G. (2014). *Mathematics in Early Childhood and Primary Education*. Research Report 18, National Council for Curriculum and Assessment, Report, Dublin. Available at http://ncca.ie/en/Publications/Reports/NCCA_Research_Report_18.pdf

- Burke, D. (2014). *Audit of Mathematics Curriculum Policy across 12 Jurisdictions*. National Council for Curriculum and Assessment, Dublin. Available at http://ncca.ie/en/Publications/Reports/Audit-mathematics-curriculum-policy.pdf

Context for change

Primary classrooms have changed a great deal since 1999. While the current mathematics curriculum is sometimes still referred to as 'new', Ireland has one of the oldest primary mathematics curricula in Europe and so it's important that we explore its suitability for the current context. Curriculum reviews and evaluations and feedback from teachers over the past decade have resulted in a call for a less 'crowded' primary curriculum that promotes collaborative learning, problem-solving approaches and supports teachers to cater for increasingly diverse needs in the classroom. Teachers have expressed concerns about meeting the challenging demands of wide-ranging and systemic factors that impact implementation such as textbooks, class size and standardised testing.

The 1999 mathematics curriculum has many strengths. With firm theoretical roots in Piagetian and radical constructivism, the curriculum promotes the development of children's meaning making, mathematical language, skills and concepts as well as fostering positive attitudes to maths. There remains, however, scope for improvement. Contemporary thinking and research offers fresh insights into 'how children learn' and 'why they learn in particular circumstances'. This thinking, which has strong Vygotskian influences promotes learning as a social and collaborative process where children's learning is enhanced through active participation, engaging in 'mathematization'[2], working collaboratively with others as well as children building positive identities of themselves as mathematicians. This shift in theoretical perspective demonstrates the need for revisiting the aims of the PMSC and identifying where improvements can be made building on the many strengths of the current curriculum.

The context for change and the development of a new primary mathematics curriculum is grounded in learning from recent research, literature, international studies, audits and national and international assessments available. The background paper aims to exemplify this learning and lay the foundations for change towards the development of the new mathematics curriculum.

The new primary mathematics curriculum will be presented using broad learning outcomes. These outcomes will replace the existing content objectives. Informed by research, the learning outcomes will describe the learning that children will be able to demonstrate at the end of a two-year period. It is intended that learning outcomes will give teachers more flexibility and opportunity to plan for, and provide rich learning experiences for children in the classroom. Progression continua, along with examples of children's mathematical learning, will support teachers to interpret and differentiate learning outcomes supporting children to learn at a level and pace appropriate to them. Furthermore, support material will help to bring to life practical ideas on effective approaches to teaching mathematics as evidenced in research.

What is mathematics?

Terms such as mathematics, numeracy, and mathematical or quantitative literacy have different meanings in different contexts, resulting *in difficulties in the debate about critical aspects of mathematical education* (Turner, 2012, p.1). Frequently there is ambiguity

[2] Mathematization involves children interpreting and expressing their everyday experiences in mathematical form and analysing real world problems in a mathematical way through engaging in key processes such as connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising (Ginsburg, 2009; Treffers and Beishuizen, 1999).

between the way people commonly use these terms and their intended meaning. Some view numeracy as more practically oriented and a part of mathematics (Dunphy *et al*., 2014), while others consider mathematics as part of numeracy, or mathematical or quantitative literacy in general (Turner, 2012). Discourse regarding terminological issues is ongoing and precise meanings continue to be debated (INTO, 2013). Of late, the Department of Education and Skills appears to favour the term 'numeracy' in various publications regarding mathematics stating that *numeracy is not limited to the ability to use numbers, to add, subtract, multiply and divide* but *encompasses the ability to use mathematical understanding and skills to solve problems and meet the demands of day-to-day living in complex social settings* (DES, 2011, p.8). Similarly, the authors of the NCCA-commissioned research reports on mathematics (Reports 17 and 18, 2014), adopt Hersh's (1997) *view of mathematics as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context* (p.xi); and, in keeping with others (e.g. Dweck, 2000; Boaler, 2009), consider that everyone is able to solve problems, communicate their mathematical thinking, and make sense of the world through mathematics. Understanding the nature of mathematics and clarifying what it means for children to engage in doing mathematics is fundamental to the development of a new PMC, and would make a good starting point for discussion. NCCA Report 17 (Dunphy *et al*., 2014, pp.33-36) provides a more detailed account regarding contemporary definitions of mathematics education.

The Primary School Mathematics Curriculum (1999)

The 1999 Primary School Mathematics Curriculum (PSMC)[3] which replaced the 1971 mathematics curriculum, views mathematics as:

> *…the science of magnitude, number, shape, space, and their relationships and also as a universal language based on symbols and diagrams. It involves the handling (arrangement, analysis, manipulation and communication) of information, the making of predictions and the solving of problems through the use of language that is both concise and accurate.* (DES, p.2)

[3] Available online at http://curriculumonline.ie/Primary

The PSMC (1999) is based on constructivist principles and comprises the following five strands for children from junior infants to sixth class: Number, Algebra, Shape and Space, Measures, and Data; with Early Mathematical Activities an additional strand for junior infants only. These strands are considered interrelated and are subdivided into various strand units. The content of the PSMC is divided into four levels or stages (infants, first and second classes, third and fourth classes, and fifth and sixth classes), delineated by year and accompanied by Teacher Guidelines[4]. The curriculum identifies the following six mathematical skills which children need to develop: Applying and Problem-Solving, Communicating and Expressing, Integrating and Connecting, Reasoning, Implementing, and Understanding and Recall; and encourages each child *to be confident and to communicate effectively through the medium of mathematics* (p.2). The PSMC promotes a wide range of teaching methodologies with cross-curricular linkage and integration. Guided-discovery learning and less reliance on textbooks and/or workbooks are encouraged. Collaborative and active learning in a mathematics-rich environment is promoted along with the use of concrete learning resources and digital technology for all classes. Discussion and the development of mathematical language are highlighted as central to children's learning of mathematics and the importance of developing estimation skills is also emphasised. Real-life problem-solving is viewed as a key element of the curriculum since it helps develop higher-order thinking skills, and highlights how mathematics can be used in everyday life. The PSMC outlines what should be assessed and offers a range of assessment practices to elicit information regarding children's progress.

The suggested time allocation for mathematics (DES, 1999) was originally 2 hours 15 minutes per week in infant classes, and 3 hours per week in all other classes, but was subsequently changed to 3 hours and 25 minutes, and 4 hours and 10 minutes respectively (Circular 0056/2011).

Since 1999, NCCA has published materials to provide additional help with the implementation of the PSMC. These materials include planning resources for teachers, a glossary of mathematical terms and bridging materials for $5^{th}/6^{th}$ classes to help children prepare for post-primary school. The NCCA also developed a suite of materials to support parents in

[4] Available online at http://curriculumonline.ie/Primary

helping their children to learn mathematics. These materials include tip sheets and videos of children and parents learning together[5].

Critique of the PSMC

In general, the PSMC was well-received by teachers and schools. The PSMC has many strengths. A recent desktop audit by Burke (2014) affirmed the comparative strength of the PSMC to the mathematics curricula in 13 other jurisdictions. Content, structure, and banding arrangements of the current curriculum were considered to be typical of international mathematics curricula. Additionally, the succinct articulation of content objectives for each of the five strands, at each of the eight class levels, were identified as a strength. Indeed, a recent report (Eivers and Clerkin, 2013) found the PSMC, while outdated, to be reasonably well aligned with the TIMSS mathematics assessment framework and items.

In a review by the NCCA (2005), Number was identified by teachers at all classes as the most useful strand, with Data (new to the PSMC) identified as least useful. Table 1 illustrates the main strengths and challenges/weaknesses with the curriculum as reported by teachers in that review.

Table 1: Strengths and challenges/weaknesses of the PSMC reported by teachers (NCCA, 2005)

Strengths	**Challenges / Weaknesses**
Children's enjoyment of mathematics Child-centred Emphasis on practical work Children's success in specific content areas	Time Appropriate use of assessment tools Catering for the range of children's abilities

[5] Available online at http://www.ncca.ie/en/Curriculum_and_Assessment/Parents/Primary/

One of the main criticisms levelled at the wider primary curriculum (1999) was the apparent disconnect between curriculum and assessment, with assessment ostensibly treated as an 'add-on' activity, and a lacuna regarding Assessment for Learning (AfL) evident in official curriculum documentation (Sugrue, 2004; 2011). While the NCCA published *Assessment in the Primary School Curriculum: Guidelines for Schools* in 2007, few teachers received CPD regarding these guidelines, and while teachers are willing to embrace assessment in their classrooms, the guidelines may remain underused (INTO, 2010).

Teachers, while acknowledging that the curriculum was flexible and had many strengths, have highlighted that it created unrealistic expectations and resulted in excessive paperwork (INTO, 2015). Additionally, they have identified the issue of curriculum overload, believing there is too much content, coupled with too many subjects, making it impossible to teach all subjects to a high standard. Furthermore, teachers believe that the curriculum can only be implemented effectively when schools are properly resourced and in receipt of high quality, practical, whole-school focussed CPD (INTO, 2015). Consequently, while these issues refer to the 1999 curriculum as a whole, they also need to be considered when developing the new PMC, perhaps through increased integration or teacher autonomy (INTO, 2015).

By international standards, Ireland's range of curriculum supports in mathematics is limited and the articulation of attainment expectations and the provision of exemplars, lags behind other countries (Burke, 2014). Furthermore, mathematics curricula in many jurisdictions have recently undergone significant redevelopment and improvement, and have incorporated relevant research, literature and contemporary thinking in mathematics and assessment (Burke, 2014), further highlighting the need for review and redevelopment.

Implementation of the PSMC

Discrepancies can exist between the intended curriculum and how it is implemented. Looney (2014) discussed the belief often held by policy-makers that problems with curriculum implementation result from teachers failing to follow the instructions they have been given, rather postulating that *curriculum aims are rarely a good guide to curriculum experiences* (p.8). Accordingly, it is widely recognised that the problem of curriculum implementation is difficult to solve (e.g. Sahlberg, 2007).

Insights from the classroom

A review conducted by the NCCA (2005) of the PSMC found teachers had prioritised focusing on specific curriculum content, increasing their use of practical work, and giving more attention to the use of mathematical language (NCCA, 2005). An evaluation of curriculum implementation (DES, 2005) highlighted challenges with methodologies and differentiation strategies employed, problem-solving, and assessment practices evident in classrooms. More recently, findings from incidental inspections (DES, 2010) have provided a snapshot of mathematics curriculum implementation in Irish primary classrooms. A total of 527 mathematics lessons were observed by the Inspectorate between October 2009 and October 2010. While findings of the overall implementation of the PSMC were mixed, learning outcomes were satisfactory in 85.4% of the lessons inspected. Many strengths in the provision of mathematics education were identified. However, the report highlighted particular challenges in teacher preparation, teaching approaches and methodologies, as well as in assessment *in an unacceptably high proportion of the mathematics lessons observed.* Moreover, only half of the children observed were enabled to work collaboratively, while ICT was used in only 30% of the lessons.

Other important insights into implementation of the PSMC are provided by *The Primary Classroom: Insights from the 'Growing up in Ireland' Study* (McCoy, Smyth and Banks, 2012). This report highlighted that 40% of children were found to spend three hours or less per week on mathematics, while 25% spend five or more hours, deducing that *some students have over 18 full days less instruction than others* (p.iii). As with other studies, teachers reported difficulties in catering for the range of children's abilities in mathematics, but despite this, generally high levels of children's engagement were reported. Another recent report, *National Schools, International Contexts* (Eivers and Clerkin, 2013), looked beyond the test scores achieved by Irish students in TIMSS (2011) and explored Irish classrooms with a mission to explain children's performances. Classroom practice was found to place a heavy emphasis on the Number strand, arguably to the detriment of core mathematical skills. Furthermore, relative to other countries, it was found that insufficient time was spent developing more complex problem-solving skills or learning key skills (Close, 2013b). The report found that Irish fourth-class children were

> *more likely than their peers internationally to work out problems with their class under their teacher's guidance,…and somewhat less likely to relate what they learned in a mathematics lesson to their everyday lives, or to take a written mathematics test* (p.93).

Teachers' experiences and views

The NCCA recently organised a series of focus groups around the country to obtain more up-to-date views regarding mathematics teaching and learning. Almost 100 teachers attended the focus group sessions at nine education centres. While a convenience, non-probability sample was utilised to locate focus group participants, member-checking by Education Centre personnel was used to ensure participants were practicing teachers.

Teaching mathematics

Focus group participants expressed the view that mathematics is an important life skill and that children need to be able to use mathematics outside the classroom in the real world and be comfortable with mathematics. They also highlighted that developing mathematical skills impacts on other areas of learning, thinking and problem-solving. Children talking about mathematics and explaining their approaches to problem-solving was highlighted as being an important classroom activity.

> *Make links to maths in the real world, it has to be relevant and purposeful.*

The importance of teachers understanding how children learn and starting teaching at a child's level, was raised. It was also emphasised that a child's ability level needs to be recognised and that within a class the ability range can be broad and may broaden as children get older.

> *[We need] emphasis on how children learn, massive upskilling required.*

The usefulness of teaching programmes such as Maths Recovery[6] was highlighted by participants. Teachers who had received Maths Recovery training felt they had a better understanding of mathematics and of how the concepts develop which gave them the pedagogical content knowledge they needed to teach effectively. It was felt that similar training should be extended to all teachers and not just to teachers working in DEIS settings.

> *Maths Recovery moves away from rote method, the method is 'understand', we show children a method that makes sense to us not to them. [A child] can do a sum mechanically but don't know how it's done conceptually. We want them to be able to do it as problem-solving not just process.*

Critical factors at play

The prevalence of workbooks and textbooks for teaching mathematics was noted and their usefulness was called into question. Participants highlighted that there is a need to make mathematics real to children and to use other learning methods such as group work, talking about mathematics and concrete objects. The usefulness of concrete materials was noted, in particular how they help children engage more with their learning. The importance of teachers being skilled in how to use concrete materials was also highlighted.

> *Maths language is very important, children can't develop this from books.*
>
> *Children learn maths better in peer groups, far more engaging and productive than text books.*
>
> *Getting kids to talk about maths is more important than filling in workbooks.*

Participants expressed some concerned views about the influence of standardised tests, the results of which are seen as having high value by parents. Concerns were expressed about whether teachers would teach to the test and the point was made that there is a conflict between the teachers' desire for the child to perform well on the test but recognition that this will impact negatively on the allocation of resources.

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[6] Maths Recovery is an intensive individualised teaching programme for low-attaining children in first class in primary school. The programme involves specialist teachers using a unique instructional approach, in addition to distinctive instructional activities and assessment procedures.

We plan around that test and Measure only gets one question so it's always left to end of the year. If it's not in the test much, it's not taught much, simple as.

The importance of parental influence was highlighted by teachers with the view that parents' own past experiences and understanding of mathematics can have a detrimental impact. Moreover, an exploration of perceptions of parental values suggested that many parents value traditional methods and believe that children should be taught as they were taught. The importance of engaging with parents was emphasised.

Parental expectations can value traditional learning of maths- workbooks, homework etc. This can be detrimental, they can be quite forceful that you are teaching them [their children] wrong.

In the main, participants highlighted that they valued the broad range of resources available including digital and online resources that can be used to engage children in mathematics.

Core maths hasn't changed, concepts and thinking etc but how we approach it has, we have fantastic opportunities to use other resources to help kids learn.

Provide us with online resources, suggested websites are fine but resources specially built for the curriculum have a huge benefit.

Points were made that a lack of confidence in teachers' own mathematical ability can impact upon how they approach teaching mathematics and that CPD and upskilling are needed to support teachers.

Teachers' attitudes to maths influence how children learn. Teachers can be fearful of maths and lack confidence, particularly substitute teachers coming in.

Sometimes teachers are afraid (those who are not confident in their maths ability) to try open-ended tasks.

More CPD or resources for self-improvement are needed.

Further challenges

Other perceptions of challenges impacting on the teaching and learning of mathematics shared by teachers in the focus groups, included the following.

- Class size was identified as a problem affecting how they teach mathematics and how children in their classrooms learn mathematics, as was the time available for teaching mathematics.
- Lack of classroom control was identified as impacting negatively on mathematics learning and teaching. It was noted that providing children with motivating and relevant learning experiences would help classroom management.
- Frustration was expressed at what teachers considered 'fads and initiatives'.
- Problems related to content strands in the curriculum were raised, such as the relative importance given to Number and Measurement.
- Others highlighted some problems children have with understanding number and moving from concrete to abstract concepts.
- A small number of comments were made emphasising the usefulness of traditional teaching methods compared to more recent, active learning approaches.

Contexts for learning

While the PSMC encouraged less reliance on textbooks, evidence suggests that mathematics planning and instruction in Irish primary classrooms is still regularly based around textbooks rather than the curriculum, with most children using textbooks on a daily basis, even in infant classes (Dunphy, 2009; Eivers *et al*. 2010). The following views offered by teachers provide some insight into this.

> *Planning with textbook is helpful for timing and managing to cover the curriculum in the time provided.*
>
> *Planning is dictated to by the book because it gives you structure. Without the book planning would take more work, you could dip in and out of books and photocopy pages for assessment but this is time consuming.*
>
> *Time is an important factor, to be innovative you need more time. It is difficult to balance exploratory, hands-on approach with constraints of curriculum overload, finishing book, getting ready for Sigma-T, must be finished by mid-May.*
>
> *…Maths book good for teacher to build confidence.*

Primary school teachers – NCCA focus groups (Autumn 2015)

Mathematics textbooks in Ireland have been criticised for including volumes of repeated practice with little difference in difficulty levels (Dooley *et al.* 2014). Moreover, the 'worked examples' which tend to predominate textbooks in the Irish context, have been criticised for being set almost exclusively in mathematical contexts rather than in real-life contexts (e.g. Delaney, 2010). Notably, two thirds of current TIMSS assessment items are embedded in applied contexts. However, Close (2013b) found that these 'real-life' questions proved difficult for fourth class children in Ireland since they have limited exposure to such questions at school. Additionally, the way problems in textbooks are primarily located in dedicated sections, and are predominantly word problems, has also been criticised.

> *Abstract word problems always left to end. Students who struggle with abstract never get to word problems or where you see it in real life.*
>
> *The textbook doesn't motivate kids, you need to [make] maths real to them … Context is everything for kids and choosing a good context can integrate with other subjects such as visual arts.*
>
> Primary school teachers – NCCA focus groups (Autumn 2015)

As part of a recent study (Eivers, Delaney and Close, 2014), three commercially available mathematics textbooks at third class level, were analysed to find out how well they aligned with the PSMC (Table 2). As textbooks have been found to often be the medium through which children experience the PSMC, it is interesting to view these results.

Table 2: Percentages of pages in three Irish pupil textbooks that cover each PSMC strand (adapted from Eivers *et al*., 2014)

% of PSMC objectives (N=70)		% of pages		
		Textbook A (N=174)	Textbook B (N=172)	Textbook C (N=156)
Number and Algebra	42.8	65.2	61.2	52.9
Shape and Space	24.3	8.3	13.5	16.5
Measures	24.3	20.1	20.5	23.4
Data	8.6	6.3	4.8	7.2

Close (2013a) suggests that teachers should be supported to move away from over-dependence on textbook activities and recommends that a repository of *good tasks* aligned with high quality professional development should be provided.

Professional development for teachers

A comprehensive programme of ongoing continuous professional development (CPD) was provided to help teachers implement the PSMC effectively (Harford, 2010; Sugrue, 2011). Opinions from research evaluating the impact of this large-scale, centralised CPD programme are somewhat mixed but significant (Murchan *et al*. 2009; Harford, 2010; Sugrue, 2011). Harford (2010) highlights that CPD in the Irish context has primarily been focused on equipping teachers to respond to curriculum change instead of the development of pedagogical approaches and reflective practice. A review by Murchan *et al*. (2009) revealed improved teacher knowledge but modest and varied implementation of the 1999 curriculum, and suggested that better identification of teacher needs prior to the CPD would have focused resources where they were most needed. They highlighted *an over-emphasis on planning rather than on creating local communities of practice per se* (p.466), and voiced concern that this model of CPD could lead to *a culture whereby teachers feel incapable of embracing reforms and adjusting professional practice without first receiving externally provided PD* (p.468). Sugrue (2011) concurred and suggested that there is *a need for more school-based CPD, and schools need to take more responsibility for the professional learning of staff* (p.803). In sharing their views of professional development (NCCA focus groups, Autumn, 2015), teachers echoed this preference for school-based CPD.

> *Two days training 'in-school' would be very useful because it would give teachers the confidence to go out and teach the new curriculum.*
>
> *Someone coming to school to help teachers be familiar with strands not with skills.*

Teachers expressed a need for support and professional development in the following areas particularly.

> *To ensure that teachers use the resources appropriately.*

Huge investment needed in CPD in maths. Useless allocating worthless resources. Caution against spending on resources without training.

Need for workshops, also videos of implementing curriculum with all the pitfalls.

Access to the CPD that DEIS schools get should be open to all teachers.

Many teachers felt that CPD should help teachers understand mathematical progression**.**

CPD is needed to show people what maths really is.

CPD needs to help teachers understand mathematical progression so that they can help students where they need it and fill the gaps.

Everything is taught in isolation, no connections are made.

Teachers are also looking for assessment tools and tools that will help them to diagnose why there is difficulty in learning and identify appropriate actions/steps to help children overcome their difficulty.

Good assessment tools are needed.

Need help in diagnosis- why a child is having difficulty and what to do to help them.

In acknowledging the potential for digital resources to enhance children's learning experiences, teachers also cautioned on the importance of CPD for purposeful use of these resources.

Digital resources have a huge benefit when teaching and learning maths but like we said, you have to know how and why to use them.

Indeed, current best practice suggests that CPD for teachers is job-embedded, sustained, collaborative, and linked to practice (Darling-Hammond and Richardson, 2009; Desmione, 2009; Guskey, 2000; O'Sullivan, 2011; Teaching Council, 2015). Therefore, the type, quality and effectiveness of CPD offered will undoubtedly impact the implementation of the new PMC, and so, the provision of CPD should be factored into the new mathematics curriculum discussions and consultations.

Context for curriculum development

Myriad factors can impact on the development of curricula, for example, political, economic, technological, and social, to name but a few; while international, national and local determinants also come into play. This section investigates the current context which will influence the development of the PMC. It explores policy developments and constraints at national and local levels and discusses how these have influenced the teaching and learning of mathematics in recent years. Furthermore, it explores the recent request by the Minister for Education and Skills, Mr Richard Bruton, TD, for the new primary mathematics curriculum to ensure that every child has an opportunity to develop the computational, and flexible and creative thinking skills that are the basis of computer science and coding.

We have witnessed substantial change in the Irish primary education system since the publication of the PSMC in 1999. The past 16 years have seen huge societal changes such as changes in the patterns of community and family life as well as rapid and unprecedented change in how children use and engage with digital and other media. These changes, among others, have and continue to have significant implications for schools. More recently, there has been an increase in the number of children with English as an Additional Language (EAL) also children learning mathematics through Irish. For many children, Irish is a second language and for some, possible a third or fourth language. These changes can present challenges for teachers during mathematics lessons, particularly regarding children's understanding and use of mathematical language. Additionally, the policy focus on inclusion means that many more children with special educational needs (SEN) are now attending mainstream schools in comparison to when the PSMC was launched. Responding to increased diversity in classrooms and supporting an extending range of children's learning needs poses challenges for teachers, the pedagogical challenge made more acute by the absence or low levels of additional support.

Meanwhile, class size in Irish primary schools remains the second highest in Europe, after England, with an average of 25 children per class in comparison to 20 children on average in other EU21 countries (OECD, 2015). This can result in difficulty for teachers when trying to engage in group work or talk and discussion or when using concrete materials during mathematics class. Additionally, it presents challenges when catering for the range of abilities present in most primary classes in the Irish context or when trying to support the learning of individual children. Notwithstanding these arguments, the recent OECD report (2015) also suggests that while smaller class size can lessen behavioural problems, there is little evidence that children's achievement is increased.

Universal Design for Learning (UDL), a research-based set of principles for curriculum development have also been devised since the publication of the PSCM. These principles promote equity of opportunity for all children and as such present a new lens for the development of curricula that addresses the challenges faced by schools in meeting the needs of an increasingly diverse school population (Meyer, Rose, Gordon, 2012). The development of the new PMC will be cognisant of the myriad factors impacting schools in Ireland currently as well as new theoretical perspectives offered in the literature. Of note and concurrent with developments in primary mathematics, there will be ongoing work in redeveloping the wider primary curriculum.

National Literacy and Numeracy Strategy, 2011-2020

Data from national and international assessments which suggested that Irish students were underperforming in mathematics were instrumental in the development of *Literacy and Numeracy for Learning and Life*, *the National Strategy to Improve Literacy and Numeracy among Children and Young People 2011-2020* (DES, 2011). This is a key policy document in the Irish context and has had significant influence on mathematics education in recent years. The *National Literacy and Numeracy Strategy* (2011) acknowledges the importance of mathematics education for all young people and presents a shared vision for numeracy for all stakeholders. It adopts a broad focus and emphasises the need to support numeracy in all curriculum areas and subjects. The strategy sets out a comprehensive set of targets and outlines actions that need to be taken in order to improve the teaching of literacy and numeracy in Irish schools, including robust self-evaluation. Regarding mathematics at primary level, some of the key targets in the strategy are:

- To promote better attitudes to mathematics among young people;
- To enable children's ability to understand, appreciate and enjoy mathematics;
- To improve mathematical language and ideas at early childhood level;

- To increase the percentage of children performing at the highest levels and decrease the percentage of children performing at the lowest levels in national assessments of mathematics by at least 5 percentage points and;
- To improve the way assessment information is used.

Since the introduction of the *National Literacy and Numeracy Strategy*, there has been an increase in the amount of time allocated to the teaching of numeracy (1 hour and 10 mins per week). Other relevant changes at national level include changing the B.Ed. programme from three to four years to allow extra time for the development of teachers' knowledge and pedagogical skills, especially in the area of numeracy. It is also hoped this will help produce reflective practitioners capable of applying current knowledge, methodologies and strategies in the teaching and learning of numeracy, as well enabling them to use ICT to support the teaching of numeracy.

Previous to the introduction of the *National Literacy and Numeracy Strategy*, schools were required to administer standardised testing at only two mandatory points with flexibility as to when they tested; at the end of first class or the beginning of second class, and at the end of fourth class or the beginning of fifth class, along with a requirement that the results of these tests be reported to parents (DES Circular 138/2006). Since the introduction of the Strategy, standardised assessments are now compulsory at three mandatory points—at the end of second, fourth and sixth classes in primary schools—and results are sent to the DES at the end of each year, reported to the Board of Management, as well as to parents (Circular 0056, 2011).

The *National Literacy and Numeracy Strategy* also highlights the importance of digital literacy. During recent focus group interviews (INTO, 2015), teachers acknowledged the benefits of using ICT as a pedagogical tool but highlighted that it should not dominate practice. However, they criticised the lack of ICT resources in classrooms, the inadequate broadband connectivity, the lack of technical support, and insufficient teacher professional development. Following a period of little investment in ICT in schools, the *Digital Strategy for Schools 2015-2020* outlines the Government's vision for the integration of ICT into schools to:

> *Realise the potential of digital technologies to enhance teaching, learning and assessment so that Ireland's young people become engaged thinkers, active learners, knowledge constructors and global citizens to participate fully in society and the economy* (p.5).

The strategy focuses on the following key themes:

- Teaching, Learning and Assessment Using ICT
- Teacher Professional Learning
- Leadership, Research and Policy
- ICT Infrastructure.

A core aim of the Strategy is to support and enable children to move beyond being passive users of technology to actively fostering creativity and ambition through technology. Some key objectives of the Strategy are that digital learning objectives should be embedded within future education policy and curriculum initiatives and that technology-assisted assessment should be promoted.

In the Strategy, the Minister for Education and Skills states that the NCCA will ensure that future curriculum specifications will incorporate clear statements of learning that focus on developing digital learning skills and the use of ICT in achieving learning outcomes at all levels of education (p.4). This is similar to what Shiel *et al.* suggested in 2014 when they highlighted the importance of paying adequate attention to the effective use of ICTs in mathematics lessons when developing and implementing a new PMC. Internationally, many countries provide interactive websites which offer myriad resources and lesson-enriching activities for teachers (Burke, 2014). Some countries, for example, Scotland, provide websites and applications that build on a gaming concept. Nevertheless, no country has yet organised its digital resources in line with grade or strand structures, thus making it time- consuming for teachers to access suitable resources.

Teachers' response to mandatory reporting of standardised test results

In the focus group sessions conducted by the NCCA (Autumn 2015), strong views were shared regarding the mandatory reporting of results to the DES, parents, and Boards of Management, with many teachers expressing concern about increased pressure, particularly those teaching second, fourth and sixth classes, to ensure their children performed well in standardised tests. Congruent with the findings of the INTO discussion papers (2013, 2015), Teachers in the NCCA focus groups believed standardised tests should reflect what they are teaching, and they considered that current tests do not take account of children's collaborative work or the needs of children with EAL. Teachers also questioned if current standardised tests are able to assess the range of problem-solving skills promoted in the PSMC.

> *Teaching to a standardised test means you neglect the development of reasoning, communication and problem solving skills.*
>
> *Test is limited and doesn't test skills. Some students guess and this is not valuable information.*

Teachers believed standardised tests have become 'high stakes' and query their usefulness. Participants felt that they don't test skills, they are not seen as diagnostic or a true reflection of children's ability or attainment. Moreover, teachers felt that children can guess and have a 'bad day', with some teachers teaching to the test.

> *Standardised tests are not useful because they are not designed to be diagnostic. Now they are used as high stakes and are not a true reflection.*
>
> *Sigma can influence planning, in fact it is the single most factor that influences our maths planning, we are definitely teaching to the sigma test.*

Teachers expressed that results of the tests, which may not be necessarily accurate, can have a detrimental knock-on impact on decisions about learning support and resource allocation.

> *Results inform learning support and affect resource allocation…Children who may be a 7 are not really a 7 and don't get resources and are not coping in class.*

Teachers felt that self-perceptions of children may be negatively impacted upon by the use of standardised testing and offered cautions in labelling children with STen scores.

> *Children say they are good or bad at maths, no grey area. Labelling themselves early on.*
>
> *I would much prefer to be able to tell a parent where their child is having difficulty and have a conversation around what can be done to help rather*

than a number ...there your child has a STEN of 6 I know you haven't a clue what it means but there you go.

Diagnostic testing was seen by teachers as something that is needed but is currently missing. Sigma-T was not seen as diagnostic and issues were identified with it and with the Drumcondra test.

There is a huge lack of diagnostic tests…would love to have a diagnostic test that you could give to children before you start.

Diagnostic is very important; this will actually have a positive impact on teaching learning planning etc.

Anxiety and stress for teachers, parents and children has been attributed to the mandatory reporting of standardised test results. Discussion findings suggest that schools engage in different practices regarding standardised testing, and that frequently the test manual is not being utilised. Moreover, teachers feel it can be difficult to explain the test to parents who don't always understand what the tests are telling them. While it would also seem that parents want their child to get a high score and are uncomfortable with their child sitting tests they are unprepared for (INTO, 2015).

When standardised testing became mandatory there was concern that the scores would be used to compare kids with each other, instead of what they were designed for. Writing S.T. score on the report has caused anxiety and trauma on the child of going down [STEN score].

There is too much emphasis on score and misinterpretation by parents.

Kids are too young to be doing these long tasks and they get very anxious. Often teachers forewarn kids and put a lot of pressure on them.

Critical to curriculum developments will be concurrent development and support in standardised testing and assessment. In this endeavour, the following quote from Shiel *et al.* (2014) is noteworthy:

The relatively large increase in performance observed in NA '14 suggest that the norms for existing school-based standardised tests may overestimate pupil performance, and hence may not be very useful for the purposes for which they are being used, such as setting school-level targets and identifying students with learning difficulties. This points to a need to benchmark performance on standardised tests used in schools against

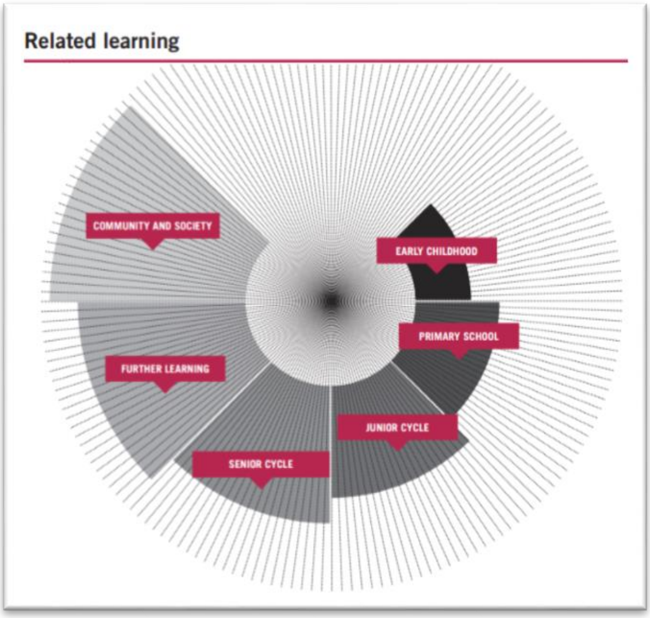
> *performance in NA '14, with a view to revising and renorming tests, perhaps in parallel with the implementation of revised curricula in English and Mathematics* (p.xv).

Transitions: The mathematics continuum

Mathematics learning and development at primary level is part of a continuum which, in the context of state-provided education, begins in the pre-school years, through primary and includes post-primary and even tertiary mathematical learning. Account must therefore be taken of the various transitions involved in children's mathematical education. The importance of children's early mathematical learning and its significance for later mathematical learning and development is now generally recognised and so a new primary mathematics curriculum will need to ensure consistency with *Aistear: the Early Childhood Curriculum Framework* (2009), thereby facilitating and supporting progression in children's learning. *Aistear* emphasises the importance of play, relationships and language for children's learning from birth to six years, and lays important foundations for children's mathematical learning in primary school. In particular, *Aistear*'s *Exploring and Thinking*, and *Communicating* themes and its integration of play as a central teaching and learning approach, help foster children's mathematical learning in the early years, and should therefore feed into the new mathematics curriculum. Further, the *National Literacy and Numeracy Strategy* (2011) recommends that teaching and learning principles and approaches in infant classes should align with those advocated in *Aistear* and acknowledges that lower adult-child ratios would be required to implement these approaches in primary classrooms.

A new PMC will also have to be cognisant of children's subsequent learning at second level and so links with the mathematics syllabus developed through Project Maths, are also important. Like the PSMC, the revised mathematics syllabus at Junior and Leaving Certificate, recognises that mathematical learning is cumulative and that each level builds on previous learning. Consequently, it should encourage learners to utilise the numeracy and problem-solving skills developed in early childhood and primary education, thus attempting to ensure connected and integrated mathematical learning and understanding across the education continuum.

Figure 1: ***Extracted from Mathematics Syllabus Leaving Certificate*** **(2012)**



The syllabus developed through Project Maths emphasises greater understanding of mathematical concepts, and the application of mathematical knowledge and skills. It encourages students to relate mathematics to everyday life and requires sense-making, problem-solving, logical reasoning, higher-order thinking skills, and engagement in rich learning activities than heretofore. To help ease students' transition from primary to post-primary, the NCCA developed a bridging framework which illustrates how the objectives of the PSMC are continued and progressed at second-level, thus ensuring continuity and progression in children's mathematical learning. The framework shows the connections between topics studied in primary and post-primary mathematics and how learning is extended.

Student achievement

Mindful that national and international test results are only one proxy for judging the effectiveness of the PSMC, it must be acknowledged that increasingly these results appear to have assumed increased importance for the government in a globalised economy. Analysis of the results of national and international comparative assessments such as the Trends in International Mathematics and Science Study (TIMSS) and the National Assessments of Mathematics and English Reading (NA) provide an objective overview of the mathematics standards of Irish primary school children, highlighting strengths and weaknesses, and also changes that take place between assessments. Periodic assessments at primary level have revealed that Irish children are underachieving in mathematics (e.g. NA'2009; TIMSS, 2011), especially in important areas of the mathematics curriculum such as problem-solving and Measures.

The 2009 National Assessments of Mathematics Achievement (NA) revealed that traditional methods of instruction still predominated Irish classrooms, with whole class teaching, children working individually rather than in pairs or groups, and the use of textbooks and workbooks very much in evidence. Measures and children's ability to apply and problem-solve proved the most difficult items at both levels while no gender differences were discovered, apart from girls' and boys' performance on Measures in sixth class. Key recommendations suggest the adoption of a stronger social constructivist perspective in mathematics teaching and learning, as well as mandatory participation in CPD. The need for more discussion, collaborative problem-solving, use of AfL in every classroom, and increased sharing of good practice at school level are also advocated. Findings from whole school evaluations (WSE) and incidental inspections (DES, 2010; Ó Donnchadha and Keating, 2013) echo many of these points (NA, 2009), once again highlighting the need for attention to assessment practices, collaborative problem-solving and opportunities to learn through talk and discussion during lessons, while also recommending greater use of differentiation and resources.

Results from the NA '14 reflect a time where there was an increased emphasis on numeracy in schools and reveal the first statistically significant improvement in children's overall mathematics since 1980 and considerably higher than in NA'09 (Shiel *at al*., 2014). These results are important since they provide data on how the *National Literacy and Numeracy* Strategy (DES, 2011) has impacted mathematics achievement and reveal if targets set out in the strategy have been achieved. Overall performance on mathematics in second and sixth classes was significantly higher in NA ’14 than in NA ‘09, with large effect sizes. There were reductions in the proportions of lower-achieving students (from 10% to 5-6%) and a small increase in the number of students performing at the higher-level. However, there is scope for students at both class levels to improve further on higher-level mathematical processes, including applying and problem-solving (Shiel, Kavanagh and Millar, 2014). The fact that there is considerable scope for improvement in mathematics in DEIS schools was also highlighted in NA ’14 by Shiel *et al*. (2014) as well as the fact that Irish children’s performance in mathematics lags behind that of literacy.

In 2011, Ireland (fourth class children only) participated in TIMSS for the first time since 1995. TIMSS provides overall achievement-related data outcomes for participating countries, thus facilitating both national and international comparisons. Additionally, TIMSS gathers other data related to the life and learning experiences of the participants such as attitudes towards school, generally and more specifically, and of particular relevance to the current paper, participants’ attitudes towards mathematics. In TIMSS 2011, Ireland was ranked 17th of 63 participating countries with a mean score of 527, above the TIMSS mathematics centre-point of 500, but significantly lower than the mean scores achieved by children in 13 other countries, including Northern Ireland and England (Eivers and Clerkin, 2012). Irish children performed strongly on Number and there were no significant gender differences in mean scores. Once again, Irish children displayed r*elative weaknesses on data display and on geometric shapes and measures* (p.27) and in the ability to reason. In comparison to TIMSS 1995, the strengths and weaknesses of Irish students remained roughly the same; there was no improvement in Ireland’s overall mean score for mathematics, but, low-achieving pupils did perform better (p.29).

Close (2013) argues that while Irish performance in TIMSS was generally satisfactory, many of the same weaknesses, highlighted in previous international studies (PISA and TIMSS) and in our own national assessments of mathematics remain and need to be addressed. Ireland has again participated in TIMSS 2015 with findings due in late 2016. TIMSS 2015 will show if Irish children's results are sustained or transferable to other contexts. An analysis of findings from the various national and international assessments discussed above highlighted that Irish children still need to improve in the area of applying and problem-solving in particular. Interestingly, Eivers and Clerkin (2013) argue that the poor performance of Irish second-level students in mathematics can be traced back to primary school mathematics, highlighting the need for any new primary mathematics curriculum to take cognisance of the new post-primary syllabus. Similarly, Close (2013b) also argues that results from TIMSS 2011 suggest *what primary pupils are taught may be at the root of the problem of Ireland's below-average standing in mathematics internationally when they move on to second level* (p.1).

Primary mathematics and its relationship with computer science and coding

The last ten years have brought unprecedented technological advances changing the way we communicate with each other, the way we access, process and manage information, and the way we ultimately think and view the world around us. Technology is now so permeated within children's everyday lives they are often referred to as 'digital natives'. What does this digital world mean though for children's learning in primary school and in particular, for their experiences with mathematics? Mathematics, like other subjects in the primary curriculum, can make an important contribution to developing children's computational, flexible and creative thinking. Such thinking is the foundation to computer science. Mathematics and computer science are complementary in so far as children's learning in mathematics will help them to develop computational thinking while computing is increasingly used in mathematics for problem-solving.

So what is computational thinking? It's a powerful thought process used to solve complex problems in schools and in the real world. Computational thinking involves taking a complex problem, understanding what the problem is and developing possible solutions which can be presented in a way that a computer or a human can understand. According to Wing (2006), computational thinking builds on the power and limits of computing processes, whether they are executed by a human or by a machine (p.33). Computational thinking involves children developing and using a number of concepts and processes including:

- logical reasoning (predicting and analysing)
- algorithms (devising steps and rules)
- decomposition (breaking down a problem into parts)
- patterns and generalisations (identifying and using simulations)
- abstractions (removing unnecessary detail)
- evaluation (making judgements).

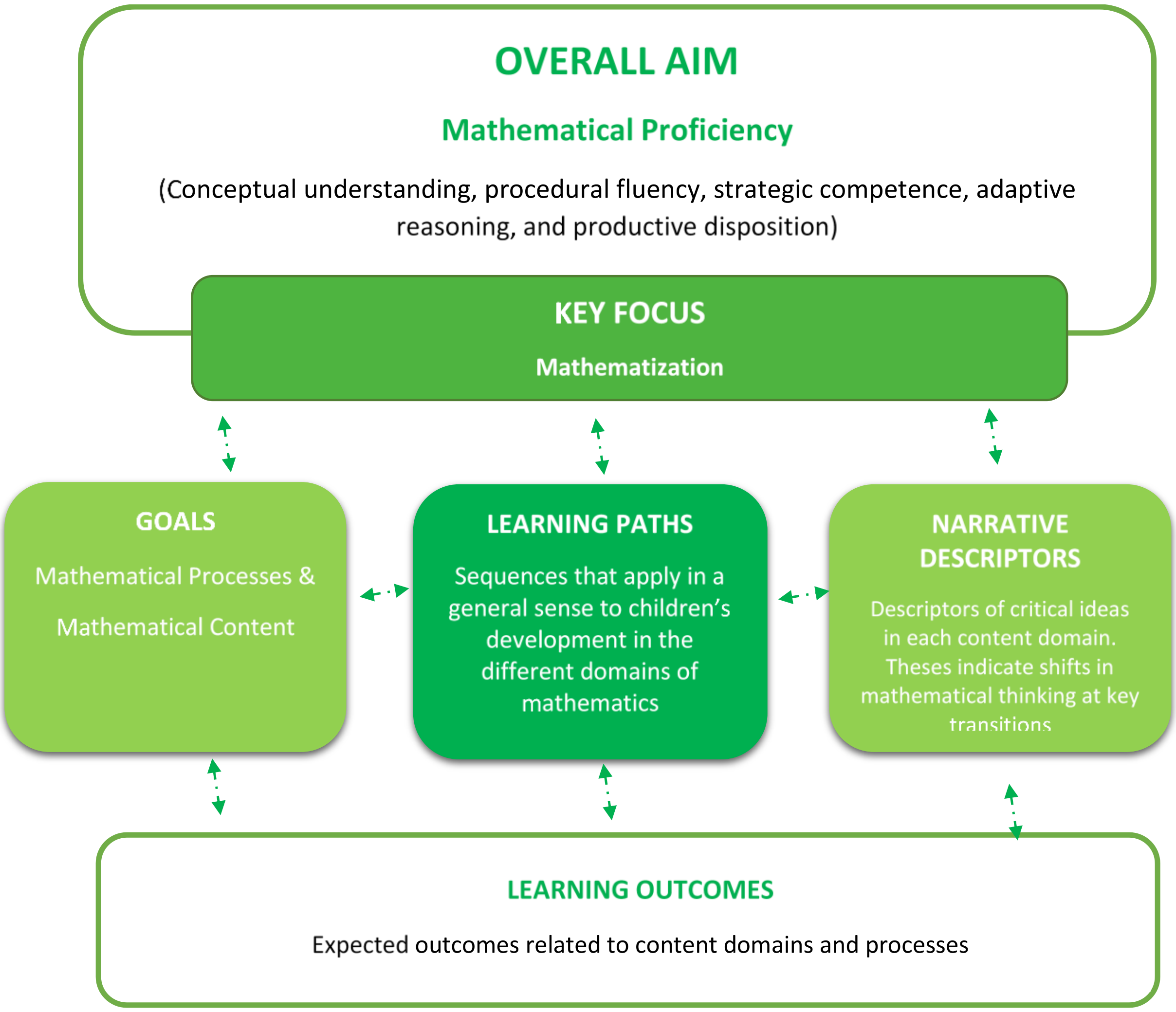
Computational thinking can be developed through playful and engaging learning experiences across the primary curriculum, for example, when writing stories children are encouraged to first plan, to think about main events and identify settings, characters, plot, etc. Or when using fair test investigations in science, children are encouraged to break the investigation down into steps, recognize patterns of what must be kept the same for each test, draw on existing understandings to reason their ideas, analyse results and draw conclusions.

Building on these foundations, computational thinking can then be further developed through rigorous and creative computer science applications such as coding. Such applications offer practical experience to children in using and extending their computational thinking as well as building the knowledge and understanding of the principles of information and computing. that leads to IT fluency. The place or significance of computer science and the extent to which its concepts, processes and applications can or should form part of the new PMC will be an important consideration in the development of the new PMC and the wider work in redeveloping the primary curriculum.

Theoretical underpinnings of a new mathematics curriculum

Research Reports 17 and 18 (Dunphy et al, 2014; Dooley et al, 2014) form a significant part of the suite of evidence used to support this background paper. Both reports are underpinned by the view that mathematics is for all and worthy of pursuit in its own right. Report 17 provides the theoretical underpinnings for the development of mathematics education in young people, and discusses current thinking and views on mathematics, specifically regarding definitions, theories, development and progression. The authors (Dunphy, Dooley and Shiel, 2014) recommend a combination of cognitive and sociocultural perspectives when envisaging a new primary mathematics curriculum (PMC). Report 18, meanwhile, deals with current thinking on the teaching and learning of mathematics. It investigates what constitutes good mathematics pedagogy and looks at appropriate structures for the development of mathematical knowledge for pre- and in-service teachers. It explores mathematical learning and development, in particular the process of mathematization. The report also discusses contemporary curricular issues and developments. Both reports suggest that the overall aim of the new mathematics curriculum should be mathematical proficiency. Mathematical proficiency consists of the five intertwined and interrelated strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (NRC, 2001). The reports recommend that since mathematization plays a pivotal role in the development of such proficiency it should permeate all mathematical teaching and learning. Additionally, the reports highlight how learning paths might be used effectively when formulating the new mathematics curriculum. Figure 2 succinctly illustrates the authors' conception of an emerging mathematics curriculum model. It includes content and process goals, learning paths and narrative descriptors, all leading to expected learning outcomes.

Figure 2*: Emerging Curriculum Model (NCCA Report 18, 2014)*



Since it is only possible to offer a brief synopsis of these two reports here, it is recommended that both reports or the executive summaries are read in full[7]. The following sections elaborate on three mathematical areas which are spotlighted in these reports, and are also emphasised in mathematics literature elsewhere. These are:

- Mathematization
- Mathematical knowledge for teaching
- Problem-solving.

[7] Available online at http://www.ncca.ie/en/Curriculum_and_Assessment/Early_Childhood_and_Primary_Education/Primary-Education/Primary_Developments/Maths/Review-and-Research/

Mathematization

The Organisation of Economic Co-operation and Development (OECD, 2002) claims that teaching students to 'mathematize' should be a primary goal of mathematics education. The term 'mathematization' was not used in the PSMC, although a number of its processes, for example, communicating, were implicit in that document. Notwithstanding, the authors of Research Reports 17 and 18 (Dunphy *et al.*, 2014; Dooley *et al.*, 2014) argue that mathematization should be central to the mathematical experience of all children.

Mathematization involves children interpreting and expressing their everyday experiences in mathematical form and comprehending the relations between abstract mathematics and real situations in the world around them (Ginsburg, 2009). This requires children to abstract, represent and elaborate on informal experiences and create models of their everyday activities. Teachers can play a critical role in facilitating children to mathematize by making meaningful connections between the mathematical strands, the real world and other areas of learning. Teachers can also assist children to mathematize by giving language to informal mathematics which children first understand on an intuitive and informal level (Clements and Sarama, 2009, p.244). For example, as a child naturally creates and extends a pattern while making a necklace with links, the teacher can effectively pose questions to encourage the child not only to use appropriate mathematical language to describe the pattern, but also to make predictions and generalisations.

Put simply, Rosales (2015) defines mathematization as *the process of understanding maths within the contexts of children's daily lives* (p.1). Enabling children to talk about their mathematical thinking (math-talk) and to engage in mathematization makes their mathematical thinking visible and helps develop their mathematical knowledge (Clements and Sarama, 2009). By modelling and fostering math-talk throughout the day, teachers can provide the math language that allows students to articulate their ideas.

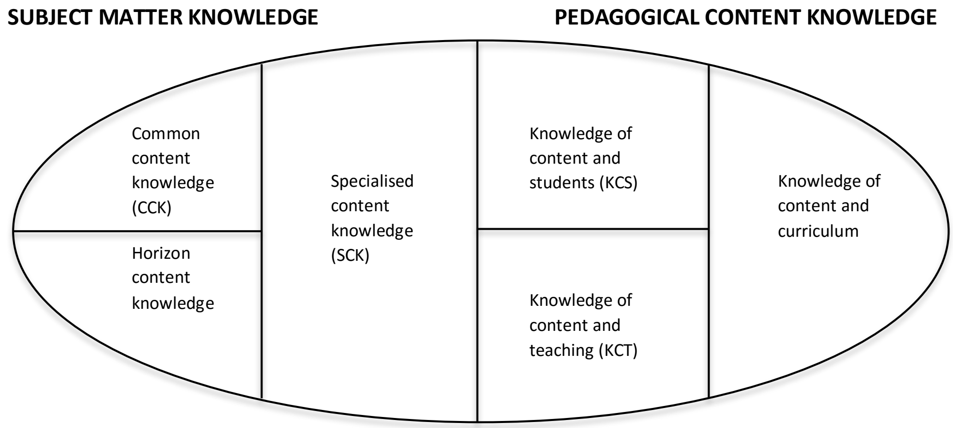
Research Reports 17 and 18 (Dunphy *et al.*, 2014; Dooley *et al.,* 2014) highlight mathematization as pivotal to the development of mathematical proficiency, and proffer that its key processes (connecting, communicating, reasoning, argumentation, representing justifying, problem-solving and generalising) should permeate any new mathematics curriculum. These processes are also core to computer science and many of its applications such as programming and coding. Mathematization strongly supports the computational skills that are also essential to proficiency in computer science and coding. Mathematization is important in building children's capacity to think flexibly and creatively and also contributes to fluency in other disciplines such as science and engineering, among others.

Dooley *et al*. (2014) highlight that mathematization takes dedicated, integrated and sustained time, and so if it is to be central to the new mathematics curriculum, significant changes in curriculum, pedagogy and curricular supports will be demanded, thus posing wide-ranging and systemic challenges. Teachers, too, will be asked to engage in mathematics teaching that is qualitatively different than what they themselves experienced. Ultimately, if teachers are to promote good mathematics learning, they must not only have an openness to and facility with the processes of mathematization, but critically, they must possess good Mathematical Knowledge for Teaching (MKT).

Mathematical Knowledge for Teaching (MKT)

Teachers' knowledge and understanding of mathematics can influence the tasks they select, their level of questioning, and how and to what extent concepts are developed within their classroom (Zopf, 2010). Using Shulman's concept of pedagogical content knowledge (PCK), Ball *et al*. (2008) specifically analysed the work of teaching from a mathematical viewpoint, and developed a theory termed mathematical knowledge for teaching (MKT) defined as …*the mathematical knowledge needed to carry out the work of teaching mathematics* which includes absolutely *everything that teachers must do to support the learning of their students, including planning, assessment, parent-teacher meetings, homework and much more* (p.395). They refined Shulman's (1986) idea of Pedagogical Content Knowledge into at least two subdomains, that of knowledge of content and students (KCS), and knowledge of content and teaching (KCT), and additionally included Shulman's idea of curricular knowledge in this section. Furthermore, they subdivided Shulman's domain of subject matter knowledge into common content knowledge (CCK) and specialised content knowledge (SCK), as well as horizon content knowledge (Figure 3).

Figure 3: Adapted from Ball *et al*., (2008)



CCK is needed by teachers and non-teachers alike, while SCK is unique to the work of teaching. Horizon content knowledge refers to an awareness of how mathematical topics are related over the span of mathematics included in the curriculum (Ball *et al.,* 2008, p.403). From this research, Ball *et al*. developed measures to assess teachers' mathematical knowledge for teaching (MKT). Delaney (2008) has adapted these measures for use in the Irish context and his research findings reveal that Irish primary teachers' levels of MKT vary substantially, with particular strengths and weaknesses (Table 3).

Table 3: Strengths and weaknesses in the MKT of Irish primary teachers (Delaney, 2010)

Strengths	Weaknesses
▪ Identifying and classifying children's mistakes ▪ Matching fraction calculations with representations ▪ Algebra	▪ Attending to explanations and evaluating understanding ▪ Identifying and applying properties of numbers and operations ▪ Matching word problems with fraction calculations

This variation in Irish primary teachers' levels of MKT is important to note since teachers frequently teach in isolation, and so, children are learning in classrooms where their teachers bring very different resources of MKT to their teaching, ultimately impacting children's learning. Teachers also need good MKT to appraise and modify mathematics textbooks (Delaney, 2010). Developing good MKT should enable teachers to provide higher quality mathematics instruction and concomitantly, increase children's achievement. Furthermore, it should enable teachers to find teaching mathematics more professionally fulfilling (Delaney, 2010), including areas they find difficult such as problem-solving.

Problem-solving

It is generally acknowledged that solving problems is vital for mathematical proficiency. Problem-solving generally refers to engagement in mathematical tasks that have the potential to provide intellectual challenges that enhance students' mathematical development (Cai and Lester, 2010). The centrality of problem-solving to mathematical learning is clear from the outset in the PSMC. The following paragraph exemplifies how problem-solving was contextualised within that document:

> *Developing the ability to solve problems is an important factor in the study of mathematics. Problem-solving also provides a context in which concepts and skills can be learned and in which discussion and co-operative working may be practised. Moreover, problem-solving is a major means of developing higher-order thinking skills. These include the ability to analyse mathematical situations; to plan, monitor and evaluate solutions; to apply strategies; and to demonstrate creativity and self-*

> *reliance in using mathematics. Success helps the child to develop confidence in his/her mathematical ability and encourages curiosity and perseverance. Solving problems based on the environment of the child can highlight the uses of mathematics in a constructive and enjoyable way.* (DES, 1999, p.8)

While the import of problem-solving was emphasised in the PSMC, evidence suggests a mismatch between what was intended and the experience of children in many Irish classrooms. An evaluation of curriculum implementation by the DES (2005) revealed *an over-reliance on traditional textbook approaches, which did not promote the development of specific problem-solving skills* (p.29). Additionally, national and international assessments and evaluations (for example, NA, 2009; TIMSS, 2011) highlighted problem-solving as an area in which Irish children continued to underperform. *Literacy and Numeracy for Learning and Life (2011),* while acknowledging that the PSMC provides clear guidance on what children should learn, also highlights weaknesses in the implementation of problem-solving approaches in Irish classrooms. It emphasises the need *to use open-ended challenging tasks that motivate young people to engage with problem-solving in a meaningful way* (2011, p.31), and suggests additional guidance should be provided for teachers on the best approaches to teaching and learning in this area. Similarly, Dooley *et al*. (2014) argue that while problem-solving is afforded a central role in the PSMC, in reality the impression given is that children first have to learn the mathematical procedures before they can apply them to practical situations, rather than problem-solving being the context in which to learn mathematics. Notwithstanding, while the PSMC and *Literacy and Numeracy for Learning and Life* (DES, 2011) highlight the importance of problem-solving for children's mathematical proficiency, neither provide details as to how problem-solving can best be implemented in the classroom context.

Research suggests that problem-solving should not be taught as a separate topic in the mathematics curriculum but rather should be an integral part of mathematics learning (Cai and Lester, 2010). Teachers need to see beyond correct or incorrect answers, and instead look at children's mathematical understanding (Kelly, 2003). Problem-solving requires a long-term approach and commitment at every class level, in every mathematical topic, and in every lesson. Teachers need to allow sufficient time for problem-solving activities, should not over-simplify the problem for their children and need to pose questions that ensure sound classroom discourse (Cai and Lester, 2010). Engaging in problem-solving activities not only helps develop children's higher-order thinking skills but also reinforces positive attitudes to mathematics.

Irish teachers' reliance on textbooks is not conducive to the development of children's problem-solving abilities, since, as Delaney (2012) highlights, many of the problems in Irish mathematics textbooks are of poor quality. He emphasises that there is little evidence to suggest that the use of problem-solving strategies, such as RUDE[8], work and proffers that the best way to learn problem-solving is through practice and the use of problems which children can approach at different levels. A popular method of solving problems is that advocated by Pólya (1945). He enunciated four basic stages in the problem-solving process:

1. Understand and explore the problem
2. Make a plan
3. Carry out the plan
4. Look back and reflect.

Problem-solving is therefore, an iterative, cyclical process. By engaging in problem-solving, children's mathematical understanding deepens. However, learners need opportunities to regularly engage in worthwhile problem-solving activities that are open-ended and connected to real-life contexts. Worthwhile problems provide a level of challenge that is intriguing and invites speculation and hard work. Problems can have multiple solutions; the solutions should not be immediately apparent but it should be possible to solve the problems within a realistic timeframe. Problems should require decision-making beyond mathematical operations and should encourage collaboration in seeking solutions. They should offer learning experiences linked to key concepts as per grade-specific curriculum expectations. Problem-solving skills can be developed in various ways, for example, through constructive play, games, puzzles, role-play, classroom situations, robotics, coding, etc. In response, the new primary mathematics curriculum could provide a repository of mathematics problems to encourage teachers to move away from textbooks and to engage in richer problem-solving activities with

[8] Read, underline, draw and estimate.

the children in their classrooms which involve looking at the real and designed world opening up great opportunities for computational thinking.

Lifelong learning in mathematics

Neale (1969) suggests that a predominant attitude to mathematics is multidimensional and includes *a liking or disliking of mathematics, a tendency to engage in or avoid mathematical activities, a belief that one is good or bad at mathematics and a belief that mathematics is useful or useless* (p.632). This definition encompasses constructs such as self-confidence, motivation, beliefs and general attitudes towards mathematics. Successive TIMSS studies have shown *a strong positive relationship within countries between children's attitudes towards mathematics and their mathematics achievement. The relationship is bidirectional, with attitudes and achievement mutually influencing each other.* (Mullis *et al.,* 2012, p.19). If children are 'good' at mathematics they are more likely to enjoy doing mathematics. This has implications for children's mathematical learning, and indeed their lifelong learning in general. The *National Literacy and Numeracy Strategy* (DES, 2011) suggests that the curriculum should not only define the knowledge and skills that children are expected to acquire in school, but also the attitudes. It emphasises that *the development of positive attitudes and motivation are vital for progression in literacy and numeracy* (p.43) and recommends the promotion of better attitudes to mathematics among children, young people and the general public. Similarly, both NCCA research reports (Reports 17 and 18, 2014) also emphasise the importance of children's attitudes and disposition to their mathematical learning and development.

In addition to helping children develop positive attitudes towards mathematics, it is also important that they develop the skill of self-regulation. Self-regulated learning (SRL) is a key characteristic of effective learning and an important skill children need to develop in order to meet the demands of 21st century learning, and ultimately lifelong learning. In general, researchers proffer that SRL includes goal-setting, motivation, metacognition (thinking about one's thinking), and the use of cognitive and metacognitive strategies (Andrade, 2013; Vrugt and Oort, 2008; Zimmerman, 2000). A growing body of evidence suggests that SRL is learnable

(Andrade, 2010; Pintrinch, 1995; Zimmerman and Schunk, 2001). Additionally, in the past decade or so, researchers increasingly suggest that SRL can be developed through the use of Assessment for Learning (AfL) practices (Andrade, 2010; Baas, Castelijns, Vermeulen, Martens and Segers, 2014; Black and Wiliam, 2009; Brookhart, 2013; Clark, 2012; Heritage, 2013; Wiliam, 2014). These experts believe that through engagement in effective AfL principles, strategies and techniques, children become more autonomous in their learning and ultimately equipped with a wide range of cognitive and metacognitive strategies, thus enabling them to self-regulate their learning.

Assessment

The centrality of assessment to inform and support good teaching and learning is widely recognised, and a combination of good Assessment for Learning (AfL) and appropriate Assessment of Learning (AoL) practices are recommended (DES, 2011; NCCA, 2007). Research suggests that using AfL on a day-to-day basis is one of the most powerful ways to improve learning in mathematics and increase children achievement (for example, Black and Wiliam, 2003; Wiliam, 2007). In the AfL literature, myriad experts mention the positive effects of using AfL on both students and teachers (Florez and Sammons, 2013; Hodgson and Pyle, 2010), and numerous reviews synthesising thousands of research studies have provided quantitative evidence of the positive impact AfL practices can have on children's learning and achievement (Black and Wiliam, 1998; Crooks, 1988; Natriello, 1987; Nyquist, 2003).

Table 4: Data from *Visible Learning* (Hattie, 2009) and *Outstanding Formative Assessment: Culture and Practice* (Clarke, 2014, p.4)

Influences on Learning	No. of Studies	Effect Size
Assessment literate students (students who know what they are learning, have success criteria, can self-assess, etc.)	209	1.44
Providing formative evaluation	30	0.90
Lesson Study	402	0.88
Classroom Discussion	42	0.82
Feedback	1310	0.75
Teacher-student relationships	229	0.72
Meta-cognitive strategies	63	0.69

Additionally, major research projects developing AfL practice have found that when teachers truly embrace AfL practices, not only is children's learning enhanced but professional and organisational learning is too (Swaffield, 2011). Furthermore, related data extracted from Hattie's (2009) synthesis of over 900 meta-analyses suggest AfL significantly impacts learning (Table 4). Regarding the role of AfL (or formative assessment) in mathematics, the National Council of Teachers of Mathematics (NCTM, 2013) in the US recently clarified their position stating:

> *Through formative assessment, students develop a clear understanding of learning targets and receive feedback that helps them to improve. In addition, by applying formative strategies such as asking strategic questions, providing students with immediate feedback, and engaging students in self-reflection, teachers receive evidence of students' reasoning and misconceptions to use in adjusting instruction. By receiving formative feedback, students learn how to assess themselves and how to improve their own learning. At the core of formative assessment is an understanding of the influence that assessment has on student motivation and the need for students to actively monitor and engage in their learning. The use of formative assessment has been shown to result in higher achievement. The National Council of Teachers of Mathematics strongly endorses the integration of formative assessment strategies into daily instruction.*

In the Irish context, the importance of regularly using AfL to enhance the teaching and learning in mathematics is also recognised by the NCCA (2007) and the DES (2011), as well as post-graduate researchers of mathematics (for example, McDonnell, 2013). However, detailed advice and support will be needed if teachers are to make effective use of AfL in teaching and learning.

Regarding Assessment of Learning (AoL) (or summative assessment), it was noted earlier that reporting standardised test results in second, fourth and sixth classes to parents and to the DES, is now mandatory. While, teachers recognise the importance and usefulness of standardised tests to aid the diagnosis of mathematical difficulties, they also have reservations regarding an over-emphasis on standardised testing, which to them represents a somewhat narrow view of learning that could negatively impact children learning and achievement (INTO, 2015).

Finally, regarding assessment in general, a criticism levelled at the PSMC was the apparent disconnect between curriculum and assessment. Therefore, it is important that the new mathematics curriculum be aligned with an assessment framework so that they can mutually support and scaffold curriculum understanding and implementation. Indeed, most countries internationally now articulate clear expectations for children's mathematical learning at specific points in their schooling and it is suggested that Ireland should follow suit.

Towards a new Primary Mathematics Curriculum

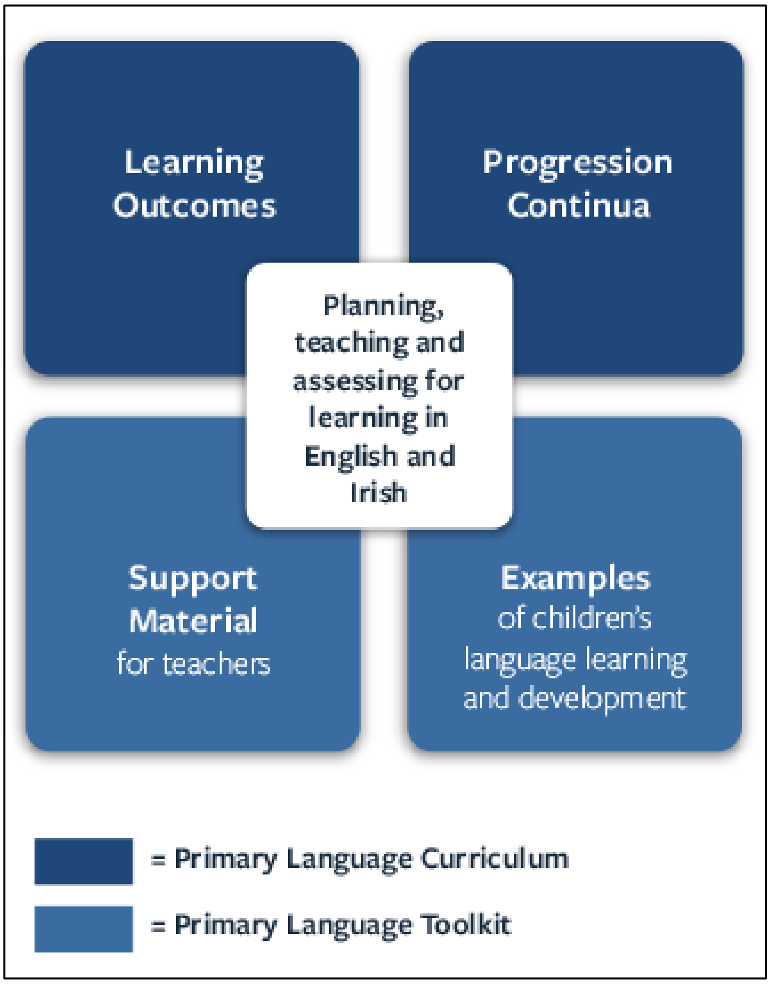
Discrepancies between the intended curriculum and the enacted curriculum are strongly evidenced by reviews of classroom implementation in the Irish context. As the enacted curriculum can be seen as a key mediating variable separating education policies from children's learning achievement (Clune, 1993; Smith and O'Day, 1991), it is critical when presenting the new primary mathematics curriculum that it not only communicates clearly the key aims and objectives of the curriculum but also supports teachers to translate the conceptual perspectives underpinning the curriculum into their own practice.

Irish teachers have expressed strong concerns about curriculum overload, leading to calls for curriculum content to be reduced and the curriculum to be re-presented as a coherent whole (INTO, 2015). Textbooks, large volumes of educational initiatives and the presentation of curriculum have exacerbated teacher's experiences of curriculum overload, among other reasons. Teachers suggested that curriculum overload can be addressed through professional autonomy and integration but acknowledge that *for integration to be successful, teachers need a very good knowledge of curriculum content* and subsequently welcome guidance around this (INTO, 2015, p.41).

Curriculum cohesion

With the dissemination of the new Primary Language Curriculum for junior infants to second class (DES, 2015), mathematics is the second area for curriculum review and redevelopment. In light of this recent development and given the importance of curriculum cohesion, it is useful to see how the Primary Language Curriculum is structured (Figure 4). The Primary Language Curriculum includes four interconnected components—Learning Outcomes, Progression Continua, Support Material and Examples of children's learning and development. Learning Outcomes describe the expected language learning and development for children at the end of a two-year period while the Progression Continua describe, in broad terms, milestones and steps in a child's journey in his/her language learning and development. Support Materials include a range of guides, podcasts and videos to support teachers' use of the Primary Language Curriculum in the school's first and second languages. The Examples of children's learning and development have been developed by teachers and children and show children's language learning and development across the three strands and across a range of school contexts.

Figure 4: The four interconnected components of the Primary Language Curriculum



Building on the work on the new language curriculum, the specification for the PMC will include the following curriculum components:

1. Introduction
2. Rationale
3. Aims
4. Strands
5. Elements
6. Expectations for learners
    a. Learning outcomes
    b. Progression continua
7. Toolkit
    a. Examples of children's learning and development

b. Support Material for teachers.

Organisation of curriculum

In considering the organisation of the new PMC, it might be useful to analyse how curricula in other jurisdictions are organised. The audit commissioned by NCCA (2014) revealed significant commonalities in how 13 countries organised their mathematics curricula. These mathematics curricula invariably included the domains of Number, Measures and Geometry, and Data and Statistics. Additionally, most included Algebra as a stand-alone strand, while some included it with Number. Table 5 compares strands from the PSMC with strands from the NCCA Audit (2014), the Jump Maths Programme[9] (Eivers *at al*. 2014), suggested strands from Report 18 (NCCA, 2014), and those from the post-primary junior cycle syllabus which is the syllabus children will be using once they transfer to second-level.

Table 5: Comparison of strands/content domains

PSMC	JUMP Maths	NCCA Report 18	NCCA Audit (No.=13)[10]	Maths Post-Primary)
1. Number 2. Algebra 3. Shape and Space 4. Measures 5. Data	1. Number Sense 2. Measurement 3. Geometry 4. Patterns and Algebra 5. Probability and Data Management	1. Number 2. Measurement 3. Geometry and Spatial Thinking 4. Algebraic Thinking 5. Data and Chance	1. Number (all) 2. Measures (all) 3. Geometry (all) 4. Data Handling and Statistics (12) 5. Algebra [stand-alone] (9) 6. Processes in Maths (5) 7. Other (2)	1. Number 2. Algebra 3. Functions 4. Geometry and Trigonometry 5. Statistics and Probability

Atweh and Goos (2011, p.223) noted that the categorisation of content into traditional mathematical fields (or strands) *may be convenient in a syllabus but it does not lend itself to dealing with real-world applications that often require cross-disciplinary approaches.* Despite the intentions of new curriculum initiatives in the US (The Common Core State Standards for

[9] . JUMP Math (Junior Undiscovered Math Prodigies) is a project co-funded by the Department of Education and Skills, Accenture, and Science Foundation Ireland. JUMP is a Canadian-designed programme intended to help children succeed at, and enjoy, learning mathematics. Information about its underlying philosophy is available at http://www.jumpmath.org/cms/

[10] Numbers in brackets indicate how many of the 13 countries audited organised and labelled strands as listed.

Mathematics, 2010) and Australia (The Australian Curriculum: Mathematics, 2012), they have been widely viewed as *lost opportunities* (Atweh and Goos, 2011; Atweh, Miller and Thornton, 2012; Hurst, 2014a). This is because curriculum content in these publications are still presented in the same linear fashion as they were in previous curriculum documents. Long lists of mathematical content that are *a mile wide and an inch deep* (Schmidt *et al.*, 2001) do little to give teachers reason to consider that mathematics may be more than unconnected bundles of information and, as a consequence, many teachers continue to teach it in the same unconnected way and inevitably, many children learn it in the same unconnected way. On discussing the inability of adults to transfer what has been learned in one situation to a different situation, Clark (2011) commented that this is because *they have been programmed to think linearly, inductively, and in little boxes* (p.34).

How children learn mathematics

Recent theories of mathematics learning have moved away from seeing learning as acquisition of knowledge towards seeing learning as the understanding of the practice of doing mathematics. This change in perspective implies the need for new learning goals for mathematics education. In supporting children's learning in mathematics, there is a strong case for balancing process and content goals. This contrasts with the design of the PSMC where content and processes are presented separately, and content is emphasised over processes. Clements, Sarama and DiBiase (2004) state that equally as important as mathematical content are general mathematical processes such as problem-solving, reasoning and proof, communication, connections, and representation; specific mathematical processes such as organising information, patterning, and composing; and habits of mind such as curiosity, imagination, inventiveness, persistence, willingness to experiment and sensitivity to patterns (p.3). Research Report 17, commissioned by the NCCA, proposes that processes and content should be clearly articulated as related goals since mathematization can be regarded as both a process and as content. For example, just as children engage in processes such as connecting, they simultaneously construct new and/or deeper understandings of content.

Mathematization goals will need to be broken down for planning, teaching and assessment purposes. This can be done through identifying critical ideas or shifts in mathematical reasoning required for the development of mathematical concepts (for example, Simon, 2006; Sarama and Clements, 2009). Such a framework provides opportunity to present children's learning as a progression towards enhanced mathematical proficiency. The specification of goals is an issue that is closely linked to pedagogy since different practices support different goals (Gresalfi and Lester, 2009) and it is acknowledged that pedagogical support will be needed to help teachers shift their thinking and practice in achieving mathematization goals.

In the classroom, children engage in mathematization by working collaboratively in groups and pairs, working on rich mathematical tasks, investigating and reasoning about problems, exploring ideas and strategies to solve these problems, and sharing and communicating their learning and thinking in a variety of ways. In providing these learning experiences for children, the teacher plays a proactive role in creating zones of proximal development where learning is scaffolded and meaning co-constructed based on awareness and understanding of the child's perspective (Bruner, 1996). Mathematization is thus contingent on a pedagogy of 'math talk', argumentation and discussion designed to support effective conceptual learning (Corcoran, 2012).

Reconceptualising content knowledge for teaching mathematics

New curriculum documents for teaching mathematics that were developed to raise standards in both Australia and the USA—the Australian Curriculum: Mathematics (ACARA, 2012) and The Common Core State Standards for Mathematics (NGA Center, 2010)—have led to much international discussion about teacher content knowledge for teaching mathematics and how mathematics should be taught (Ireland - Delaney, 2010; Australia - Callingham *et al.*, 2011, and Clarke, Clarke and Sullivan, 2012; New Zealand - Anakin and Linsell, 2014; and USA - Thanheiser *et al.*, 2013, and Green, 2014). Rather than being concerned with the amount of mathematical knowledge needed by primary teachers, some researchers (Hill and Ball, 2004, cited in Clarke, Clarke and Sullivan, 2012) suggest it may be more appropriate for policy makers to consider *how* the knowledge is held.

Recently, researchers (Charles, 2005; Clarke, Clarke and Sullivan, 2012; Siemon, Bleckley and Neal, 2012) have suggested that presenting mathematical content from the perspective of the *foundational concepts* of mathematics is key to developing teachers' mathematical content knowledge and their capacity to respond effectively to curriculum documents. Such a focus would enable teachers to make use of the many connections and links within and between such *foundational concepts* and to make them explicit to students.

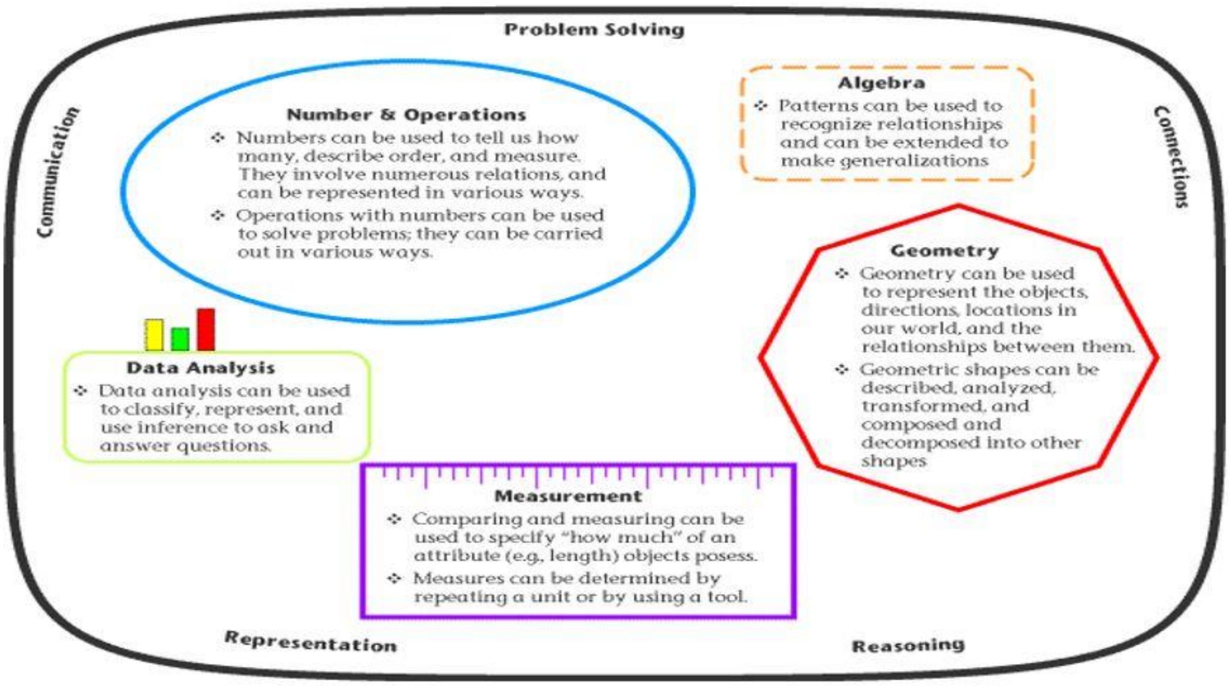
The way one views mathematics is not inconsequential and has been linked to success in mathematics. Boaler (2012) observed that people who make connection within mathematics and see it as a connected subject tend to do well in mathematics, whereas people who see mathematics as a bundle of isolated topics tend not to do so well. Presenting mathematical content and processes in terms of foundational concepts stresses the importance of conceptual understanding as the building blocks to scaffolding 'Big Ideas' in mathematics. Moreover, this presents an opportunity to support teachers to reconceptualise their ideas about mathematics teaching and learning as well as the development of their pedagogical content knowledge.

Making the case for 'Big Ideas'

The National Council of Teachers of Mathematics (NCTM) claim that teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise (NCTM, 2000, p.17). In research studies, where teaching and learning in maths was found to be most successful, teachers' mathematical content knowledge and teaching practices were anchored around a set of 'Big Ideas' in mathematics which enabled students to develop a deeper understanding of mathematics (Ma, 1999; Stigler, 2004; Weiss, Heck and Shimkus, 2004; Charles, 2005).

The notion of 'Big Ideas' of mathematics has been afforded prominence within the literature in recent time (Clements, Sarama, and DiBiase, 2004; Charles, 2005; Clarke, Clarke and Sullivan, 2012; Siemon, Bleckley and Neal, 2012) though it is still considered an elusive term. Clements and Sarama (2009) equate learning goals as the big ideas of mathematics. These big ideas are clusters of concepts and skills that are mathematically central and coherent, consistent with children's thinking and generative of future learning. For example, one 'Big Idea' is that counting can be used to find out how many there are in a collection, another would be, geometric shapes can be described, analysed, transformed and composed and decomposed into other shapes. 'Big Ideas' are the foundations of children's learning compounded by the notion that the degree of understanding is determined by the number and strength of the connections (Hiebert and Carpenter, 1992, p.67) and furthermore that we understand something if we see how it is related or connected to other things we know (Hiebert *et al.*, 1997, p.4). Charles (2005) contends that 'Big Ideas' are important because they enable us to see mathematics as a coherent set of ideas that encourage a deep understanding of mathematics, enhance transfer, promote memory and reduce the amount to be remembered (p.10). When one understands 'Big Ideas', mathematics is no longer a set of unconnected bundles of content and skills (see Table 6). Put simply, 'Big Ideas' help children to make connections with their learning in mathematics and effective teaching helps to make these connections explicit (Charles, 2005).

Table 6. Example of 'Big Ideas' underpinning different mathematical strands (Clements, Samara and Di Biase, 2004)



Reported benefits of adopting a 'Big Ideas' approach include:

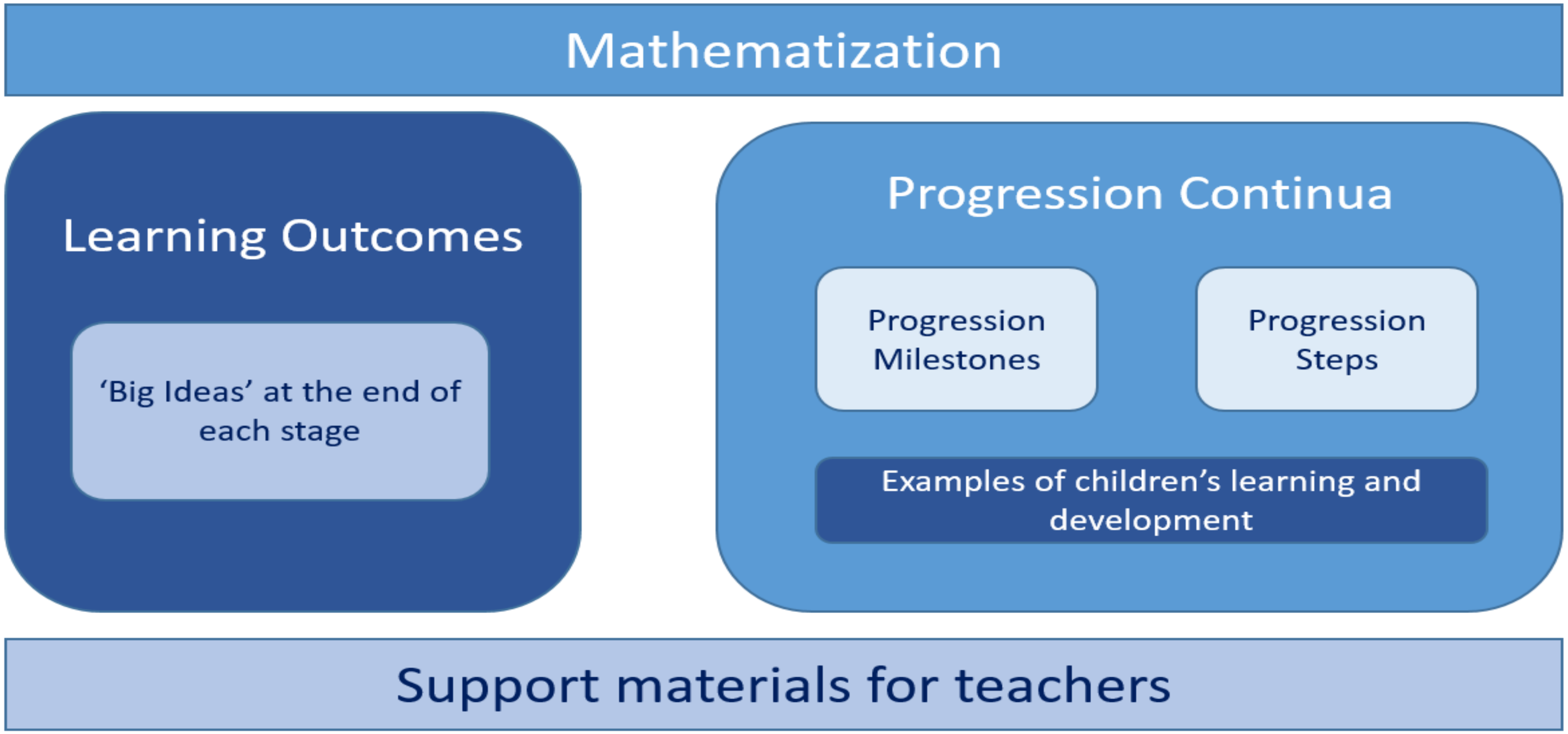
- Promotes understanding (Charles, 2005; Reys, 2008)

- Promotes memory, motivation, transfer and the development of autonomous learners (Lamdin, 2003)
- Thins an overcrowded curriculum (National Curriculum Board, 2009; National Mathematics Advisory Panel, 2008)
- Increases the number and strength of the connections that are made to other ideas and strategies (Charles, 2005)
- Supports further learning and problem-solving (Siemon, 2007; AAMT, 2009)
- Maximises progress for all by targeting teaching to key ideas and strategies (Siemon *et al.*, 2006)
- Provides curriculum coherence and articulates the important mathematical ideas that should be the focus of curriculum (Charles, 2005).

There is not necessarily any one particular way in which content ideas can be linked around 'Big Ideas' or even how these links might be presented. Hence, how 'Big Ideas' thinking can be incorporated into the curriculum will need deliberation. Notwithstanding, the literature suggests that a focus on 'Big Ideas' with their myriad links and connections would greatly enhance pedagogies for delivering mathematics curricula (Hurst, 2014). Such deep and connected knowledge would be likely to lead to more effective concept-based teaching rather than a reliance on teaching procedures. 'Big Ideas' give a new perspective to curriculum development that shows strong potential for supporting teachers in negotiating curriculum intentions, promoting a more connected view of mathematics as well as offering promise in 'thinning out' an overcrowded curriculum (Siemon, Bleckley and Neal, 2012).

The following model outlines a conceptual framework for development of the new primary mathematics curriculum aligned with NCCA curriculum specifications. This model illustrates how the relationships between the different curriculum components may be conceptualised in a new curriculum specification for primary mathematics. The model is an adaptation of the emerging curriculum model offered in Research Report No.18, (Dooley *et al.*, 2014).

Figure 1: A developing curriculum model



Brief for the development of a new Primary Mathematics Curriculum

The background paper, as evidenced by research, teacher voices and new perspectives in mathematics both nationally and internationally, has signposted the need to reconceptualise approaches to teaching, learning and assessment of mathematics for primary school children. Moreover, the background paper offers perspectives on presenting the primary mathematics curriculum in a new way that emphasises depth of learning, understanding and application of mathematical concepts and supports children to develop positive dispositions to mathematics. The following brief reflects key implications for the development of the new primary mathematics curriculum arising from the background paper as well as from the NCCA Research Reports 17 and 18 (2014).

Guiding principles

The following guiding principles offer direction and focus for the development of the new primary mathematics curriculum. Curriculum developments should aim to:

1. Reconceptualise a new curriculum to reflect new aims, learning goals and emphases

A fresh and coherent vision (blueprint) for children's learning in mathematics is necessary to guide the development of the new primary mathematics curriculum. The curriculum should be coherent in terms of aims, goals relating to both processes and content, and pedagogy. Mathematical proficiency as defined in the US context—conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (National Research Council [NRC], 2001)—provides a good starting point for the development of aims for the new PMC in the Irish context. Notwithstanding, the aims of the new primary mathematics curriculum will need to be re-contextualised and redefined for the Irish context and recognise the development of mathematical proficiency as building on pre-school and home experiences of learning mathematics as promoted within *Aistear: the Early Childhood Curriculum Framework* (2009).

The structure and presentation of the new primary mathematics curriculum will require careful deliberation and planning so as to amplify new emphases. Key emphases in the new primary mathematics curriculum will include conceptual development, mathematization, problem-solving, application of knowledge, teaching 'Big Ideas' and fostering positive dispositions to mathematics. Big ideas are a departure from a view of mathematics as a set of disconnected concepts, skills, facts and procedures and rather serve to foster integration and facilitate children to make connections within their learning in mathematics as well as other contexts. The curriculum will promote authentic application of mathematical content, ideas and skills within appropriate and relevant contexts, such as real-life situations and children's play.

Children will engage with foundational concepts in mathematics organised according to the five content domains – Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance. Early Mathematical Activities will be integrated into these five content areas. Mathematical processes such as communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting, will be foregrounded in curriculum documentation through the articulation of related mathematization goals (critical ideas). Critical ideas will indicate shifts or milestones in children's mathematical development in each foundational concept across stages, for example, two years. Critical ideas will function to support teachers to help make children's learning visible and present children's learning as a progression towards 'Big Ideas'. Narrative descriptors of mathematical content and processes will indicate progression steps in children's understanding and application of mathematical (foundational) concepts. These learning paths and narrative descriptors will be broadly specified and will outline the journey towards achieving learning outcomes. Moreover, they will serve as reference points for teachers in their planning, teaching and assessment.

2. Support children to meet the demands of 21st century learning and life

The new primary mathematics curriculum will recognise the role of early mathematical learning as a vital life skill and a foundation for citizenship in the 21st century. It will nurture the fundamental skills of conceptual development, critical reasoning, analytical thinking and problem-solving. Moreover, it will lay the foundations for children to acquire the basic language structures and foundational concepts in mathematics to enable them to interact, understanding and conceptualise the world around them. The new primary mathematics curriculum will aim to support young children to acquire a set of skills and competencies in order to meet the demands of 21st century learning and life, to create new knowledge and to navigate their way through change, uncertainty and opportunity.

3. Ensure continuity and progression across sectors

Work on the new mathematics curriculum will take cognisance of developments at both early childhood (*Aistear*) as well as at junior cycle in order to ensure continuity and progression across sectors. The development of the mathematics curriculum for junior infants to second class, in particular, will need to build on and align with the pedagogical emphases in *Aistear*.

A common language for communicating curriculum goals and principles will need to be established to facilitate cross-sectoral communication and transitions so that parents and educators across early childhood settings can communicate about children's mathematical experiences and the features of pedagogy that support children's learning.

4. Support and build understanding and application of 'Big Ideas' in mathematics

Learning outcomes will describe the expected learning and development for children at the end of a stage in terms of critical ideas in mathematics. Critical Ideas will indicate shifts in children's mathematical development towards understanding and applying the 'Big Ideas' in mathematics. The progression continua will describe, in broad terms, children's mathematical learning and thinking towards these 'Big Ideas'.

Key starting points for the development of an outcome-focused curriculum might be:

- Defining 'Big Ideas' – Drawing on research, what are the 'Big Ideas' we want children to understand and how should we present this within the context of the curriculum?
- Identifying desired results according to stages of learning – For each 'Big Idea', what are the critical ideas we want children to understand and use by the end of each stage of learning?
- Planning learning experiences – What foundational concepts and learning activities will facilitate understanding of the 'Big Ideas'?
- Determining assessment evidence – How will we know children have understood the 'Big Ideas'?

Similarly, planning and assessment approaches will be aligned with learning outcomes and progression milestones. Learning outcomes or critical ideas will serve as starting points for planning, teaching and assessing children's mathematical learning. Progression milestones and steps will further scaffold the planning, teaching and assessment processes. For the purpose of supporting progression in children's mathematical learning and development, support materials will be provided to offer multiple, diverse and appropriate opportunities for children to demonstrate learning and achievement.

5. Promote the principles of inclusion, equity and access

The curriculum will be developed in line with the principles of universal design for learning and as such, promote the principles of equity and access for children with a diverse range of abilities. For children with special educational needs and in particular, those with severe and profound and low moderate needs, the curriculum will outline what is appropriate and relevant for them to know and provide differentiated support so they can access this learning. The curriculum will support children who attend Irish- and English-medium schools, and acknowledge and support children from different language backgrounds where neither English nor Irish is their first language. It will be considerate of the wide range of diverse backgrounds that children come from and their differing starting points as they enter primary school, including children from socio-economically disadvantaged backgrounds.

Furthermore, the curriculum will support teachers to recognise children's development in mathematical conceptual understanding and application, and decide how this can be extended further through mathematical experiences.

6. Outline changes in pedagogy and curriculum supports

While foregrounding mathematical proficiency as the aim of the mathematics curriculum has the potential to change the kind of learning that children experience in primary schools, it also demands significant changes in pedagogy and necessitates curriculum supports to scaffold this change. The curriculum should inform teachers about goals, learning paths and critical ideas in developing understanding around the 'Big Ideas' of mathematics. Accordingly, teachers should be encouraged and enabled to develop content knowledge and pedagogical knowledge for teaching primary mathematics.

Given the complexities involved, teachers will require appropriate support material to develop the knowledge, skills, and dispositions required to teach mathematics well. Support material will draw on research and practice to provide teachers with practical support in using a range of pedagogies evidenced in research as being effective in mathematical teaching and learning.

Support material might include:

- Lesson Study or research lessons focused on connecting practice and 'Big Ideas' to allow teachers to interrogate and negotiate the new primary mathematics curriculum with colleagues as it relates to their setting and content.
- Video tutorials, for example, on initiating the planning process.
- A bank of rich tasks devised by teachers and linked to learning outcomes and the development of foundational concepts.

7. Address the need for appropriate resources to support teaching, learning and assessment including the promotion of digital learning and technology

In promotion of the centrality of mathematical proficiency and 'Big Ideas' within the new primary curriculum, particular consideration will need to be taken in addressing the issue of over-reliance on traditional textbooks. Curriculum developments will need to address the need for appropriate resources to support the teaching, learning and assessment of the curriculum. Collaborative and active learning in a mathematics-rich environment, along with the use of concrete learning resources and digital technology for all classes, will be embedded in curriculum material.

Curriculum supports will exemplify how tools, including digital tools, can scaffold and enhance learning and assessment. Support material developed in line with the curriculum should also aim to deepen children's mathematical understanding, provide a level of challenge, be open-ended and connected to real-life contexts.

8. Consider the role of external factors, wide-ranging and systemic challenges impacting curriculum implementation

External factors outside the curriculum development space but nonetheless significantly impacting on effective curriculum implementation, will need to be considered in the development of the new primary mathematics curriculum. As noted in the background paper, amongst these factors are

- Standardised testing
- Textbooks
- Curriculum dissemination and professional development
- Messaging across the system, including open communication and dialogue with parents and the wider community focusing on the importance of mathematics learning in the early years, the goals of the mathematics curriculum and ways in which children can be supported to achieve these goals.

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An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

Research Report No. 17

Mathematics in Early Childhood and Primary Education (3–8 years)

Definitions, Theories, Development and Progression

Elizabeth Dunphy, Thérèse Dooley and Gerry Shiel

With Deirdre Butler, Dolores Corcoran, Miriam Ryan and Joe Travers
International Advisor: Professor Bob Perry

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An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

ISSN 1649-3362

National Council for Curriculum and Assessment
35 Fitzwilliam Square, Dublin 2.

T: +353 1 661 7177
F: +353 1 661 7180
E: info@ncca.ie

www.ncca.ie

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Elizabeth Dunphy, St. Patrick's College, Dublin

Thérèse Dooley, St. Patrick's College, Dublin

Gerry Shiel, Educational Research Centre, Dublin

Deirdre Butler, St. Patrick's College, Dublin

Dolores Corcoran, St. Patrick's College, Dublin

Miriam Ryan, St. Patrick's College, Dublin

Joe Travers, St. Patrick's College, Dublin

Professor Bob Perry, Charles Sturt University, Australia

Research conducted on behalf of the
National Council for Curriculum and Assessment

2014

Acronyms

AAMT	Australian Association of Mathematics Teachers
Aistear	The Early Childhood Curriculum Framework (2009)
CCSSM	Common Core States Standards for Mathematics (United States)
CHAT	Cultural historical activity theory
DEIS	Delivering Equality of Opportunities in Schools
DES	Department of Education and Skills (formerly Department of Education and Science)
DfEE	Department for Education and Employment (United Kingdom)
EAL	English as an Additional Language
ECA	Early Childhood Australia
ENRP	Early Numeracy Research Project (Victoria, Australia)
ERC	Educational Research Centre
HLT	Hypothetical Learning Trajectory
ICT	Information and Communication Technology
KDU	Key Developmental Understanding
LFIN	Learning Framework in Number (Wright, Martland & Stafford, 2006)
LT	Learning Trajectory
NAEYC	National Association for the Education of Young Children (United States)
NCCA	National Council for Curriculum and Assessment
NCTM	National Council of Teachers of Mathematics (United States)
NGA	National Governors Association (United States)
NRC	National Research Council (United States)
OECD	Organisation for Economic Cooperation and Development
PISA	Programme for International Student Assessment
PM	Project Maths
PSC	Primary School Curriculum (1999)
PSMC	Primary School Mathematics Curriculum (1999)
RME	Realistic Mathematics Education
RTI	Response to Intervention (United States Initiative)
STEM	Science, Technology, Engineering and Mathematics
TAL	Tussendoelen Annex Leerlijinen (in Dutch); Intermediate Attainment Targets (in English)
TIMSS	Trends in International Mathematics and Science Study

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The executive summaries of reports No. 17 and No. 18 are available online at *ncca.ie/primarymaths*. The online versions include some hyperlinks which appear as text on dotted lines in this print copy.

Acknowledgements

The authors thank the National Council for Curriculum and Assessment for commissioning and supporting this report. We are very thankful to Arlene Forster and Aoife Kelly of the NCCA for providing detailed feedback on earlier drafts of the report. The authors are also indebted to Professor Bob Perry, Charles Sturt University, Australia who read early drafts of the report and who provided expert advice on various issues addressed in the report.

Executive Summary

The review of research on mathematics learning of children aged 3–8 years is presented in two reports. These are part of the NCCA's Research Report Series (ISSN 1649–3362). The first report (Research Report No. 17) focuses on theoretical aspects underpinning the development of mathematics education for young children. The second report (Research Report No. 18) is concerned with related pedagogical implications. The key messages from Report No. 17 are presented in this Executive Summary.

A View of Mathematics

Both reports are underpinned by a view of mathematics espoused by Hersh (1997). That is, mathematics as 'a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context' (p. xi). Mathematics is viewed not only as useful and as a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004). All children are viewed as having an ability to solve mathematical problems, make sense of the world using mathematics, and communicate their mathematical thinking.

Context

The context in which this report is presented is one in which there is a growing awareness of the importance of mathematics in the lives of individuals, in the economy and in society more generally. In parallel with this there is a growing realisation of the importance of the early childhood years as a time when children engage with many aspects of mathematics, both at home and in educational settings (Ginsburg & Seo, 1999; Perry & Dockett, 2008). Provision for early childhood education in Ireland has also increased. A recent development is free preschool education for all children in the year prior to school entry. In addition, a new curriculum framework, *Aistear* (National Council for Curriculum and Assessment [NCCA], 2009a; 2009b), is available to support adults in developing children's learning from birth to six years. At the same time, however, there are concerns about the levels of mathematical reasoning and problem-solving amongst school-going children, as evidenced in recent national and international assessments and evaluations at primary and post-primary levels (e.g., Eivers et al., 2010; Perkins, Cosgrove, Moran & Shiel, 2012; Jeffes et al., 2012). While the 1999 Primary School Mathematics Curriculum (PSMC) has been well received by teachers (NCCA, 2005), the Inspectorate of the then Department of Education and Science identified some difficulties with specific aspects of implementation (DES, 2005). The current report envisions a revised PSMC that is responsive to these concerns, that recognises the importance of building on children's early engagement with mathematics, and which takes account of the changing demographic profile of many educational settings, and the increased diversity among young children.

Definitions of Mathematics Education

Current views of mathematics education are inextricably linked with ideas about equity and access and with the vision that mathematics is for all (Bishop & Forganz, 2007), i.e. all children should have opportunities to engage with and benefit from mathematics education and no child should be excluded.

Mathematics education is seen as comprising a number of mathematical practices that are negotiated by the learner and teacher within broader social, political and cultural contexts (Valero, 2009). An interpretation of mathematics that includes numeracy but is broader should underpin efforts towards curricular reform in Ireland. This report identifies mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition) (NRC, 2001) as a key aim of mathematics education. It is promoted through engagement with processes such as connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. All of these are encompassed in the overarching concept of mathematization. This involves children interpreting and expressing their everyday experiences in mathematical form and analysing real world problems in a mathematical way through engaging in these key processes (Ginsburg, 2009a; Treffers & Beishuizen, 1999). Thus mathematization is identified as a key focus of mathematics education and as such it is given considerable attention in this report. Mathematics education should address the range of mathematical ideas that all children need to engage with. It should not be limited to number.

Theoretical Perspectives

Cognitive and sociocultural perspectives provide different lenses with which to view mathematics learning and the pedagogy that can support it (Cobb, 2007). Cognitive perspectives are helpful in focusing on individual learners while sociocultural perspectives are appropriate when focusing on, for example, pedagogy (Cobb & Yackel, 1996). Sociocultural, cognitive perspectives and constructionism all offer insights which can enrich our understanding of issues related to the revision of the curriculum. They do so by providing key pointers to each of the elements of learning, teaching, curriculum and assessment. Used together they can help in envisaging a new iteration of the PSMC.

In this report, learning mathematics is presented as an active process which involves meaning making, the development of understanding, the ability to participate in increasingly skilled ways in mathematically-related activities and the development of a mathematical identity (Von Glasersfeld, 1984; Rogoff, 1998; Lave & Wenger, 1991). Learning also involves the effective use of key tools such as language, symbols, materials and images. It is seen to be supported by participation in the community of learners engaged in mathematization, in small-group and whole class conversations.

The proactive role of the teacher must be seen to involve the creation of a zone of proximal development, the provision of scaffolding for learning and the co-construction of meaning with the child based on awareness and understanding of the child's perspective (e.g., Bruner, 1996). It also involves a dialogical pedagogy of argumentation and discussion designed to support effective conceptual learning and the ability for teachers to act contingently (e.g., Corcoran, 2012).

Language and Communication

Cognitive/constructivist and sociocultural perspectives on learning emphasise the key role of language in supporting young children's mathematical development. Emerging learning theories point to the importance of mathematical discourse as a tool to learn mathematics (e.g., Sfard, 2007). In addition to introducing young children to mathematical vocabulary, it is important to engage them in 'math talk' – conversations about their mathematical thinking and reasoning (Hufferd-Ackles, Fuson & Sherin, 2004). Such talk should occur across a broad range of contexts, including unplanned and planned mathematics activities and activities such as storytelling or shared reading, where mathematics may be secondary. Children at risk of mathematical difficulties, including those living in disadvantaged circumstances, may need additional, intensive support to develop language and the ability to participate in mathematical discourse (Neuman, Newman & Dwyer, 2011).

Research indicates an association between the quality and frequency of mathematical language used by carers, parents and teachers as they interact with young children, and children's development in important aspects of mathematics (Klibanoff et al., 2006; Gentner, 2003; Levine et al., 2012). This highlights the importance of adults modelling mathematical language and encouraging young children to use such language. Conversations amongst children about mathematical ideas are also important for mathematical development (e.g., NRC, 2009).

Defining Goals

The goal statements of a curriculum should be aligned with its underlying theory. Curriculum goals should reflect new emphases on ways to develop children's mathematical understandings and to foster their identities as mathematicians (Perry & Dockett, 2002; 2008). This report proposes that processes and content should be clearly articulated as related goals (e.g., mathematization can be regarded as both a process and as content since as children engage in processes e.g., connecting, they construct new and/or deeper understandings of content). This contrasts with the design of the Primary School Mathematics Curriculum (PSMC), where content and processes are presented separately, and content is emphasised over processes. An approach in which processes are foregrounded, but content areas are also specified, is consistent with a participatory approach to mathematics learning and development.

General goals need to be broken down for planning, teaching and assessment purposes. This can be done through identifying critical ideas i.e., the shifts in mathematical reasoning required for the development of mathematical concepts (e.g., Simon, 2006; Sarama & Clements, 2009). An understanding of this framework enables teachers to provide support for children's progression towards curriculum goals.

The Development of Children's Mathematical Thinking

The idea of stages of development in children's mathematical learning (most often associated with Piaget) has now been replaced with ideas about developmental/learning paths. This is a relatively recent area of research in mathematics education (Daro et al., 2011) and as such is still under development. Learning paths are also referred to as learning trajectories. They indicate the sequences that apply in a general sense to development in the various domains of mathematics (e.g., Fosnot & Dolk, 2001; Sarama & Clements, 2009; van den Heuvel-Panhuizen, 2008). This report envisages that general learning paths will provide teachers with a basis for assessing and interpreting the mathematical development in their own classroom contexts, and will lead to learning experiences matched to individual children's needs.

There is variation in the explication of learning paths, for example, linear/nonlinear presentation, level of detail specified, mapping of paths to age/grade, and role of teaching. Different presentations reflect different theoretical perspectives. An approach to the specification of learning paths that is consistent with sociocultural perspectives is one which recognises the paths as

i. provisional, as many children develop concepts along different paths and there can never be certainty about the exact learning path that individual children will follow as they develop concepts

ii. not linked to age, since this suggests a normative view of mathematics learning

iii. emerging from engagement in mathematical-rich activity with children reasoning in, and contributing to, the learning/teaching situation (e.g., Fosnot & Dolk, 2001; Stigler & Thompson, 2012; Wager & Carpenter, 2012).

Assessing and Planning for Progression

Of the assessment approaches available, formative assessment offers most promise for generating a rich picture of young children's mathematical learning (e.g., NCCA, 2009b; Carr & Lee, 2012). Strong conceptual frameworks are important for supporting teachers' formative assessments (Carr & Lee, 2012; Ginsburg, 2009a; Sarama & Clements, 2009). These influence what teachers recognise as significant learning, what they take note of and what aspects of children's activity they give feedback on. There is a range of methods (observation, tasks, interviews, conversations, pedagogical documentation) that can be used by educators to assess and document children's mathematics learning and their growing identities as mathematicians. Digital technologies offer particular potential in this regard. These methods are challenging to implement and require teachers to adopt particular, and for some, new, perspectives on mathematics, mathematics learning and assessment. Constructing assessments which enlist children's agency (for example, selecting pieces for inclusion in a portfolio or choosing particular digital images to tell a learning story) has many benefits. One benefit is the potential for the inclusion of children's perspectives on their learning (Perry & Dockett, 2008).

In the main, the current literature affords scant support for the use of standardised tests with children in the age range 3–8 years (e.g., Mueller, 2011). More structured teacher-initiated approaches and the use of assessment within a diagnostic framework may be required on some occasions, for example, when children are at risk of mathematical difficulties. However, research indicates a range of factors problematising the use of standardised measures with young children (e.g., Snow & Van Hemel, 2008).

The complex variety of language backgrounds of a significant minority of young children presents a challenge in the learning, teaching and assessment of mathematics. Children for whom the language of the home is different to that of the school need particular support. That support should focus on developing language, both general and mathematical, to maximise their opportunities for mathematical development and their meaningful participation in assessment (Tabors, 2008; Wood & Coltman, 1998). Educators carrying out assessment procedures such as interviews, observations or tasks in an immersion context have the dual purpose of assessing and evaluating both the mathematical competences and language competences of the child, to gain a full picture. Dual language assessment is particularly desirable in this context (Murphy & Travers, 2012; Rogers, Lin & Rinaldi, 2011).

Addressing Diversity

Mathematics 'for all' implies a pedagogy that is culturally sensitive and takes account of individuals' ways of interpreting and making sense of mathematics (Malloy, 1999; Fiore, 2012). An issue of concern is the limitations of norms-based testing which can disadvantage certain groups. This indicates the need to use a diverse range of assessment procedures to identify those who are experiencing learning difficulties in mathematics.

The groups of individuals that often require particular attention in the teaching and learning of mathematics are 'exceptional' children (those with developmental disabilities or who are especially talented at mathematics) (Kirk, Gallagher, Coleman, & Anastasiow, 2012). These individuals do not require distinctive teaching approaches, but there is a need to address their individual needs. In particular, the use of multi-tiered tasks in which different levels of challenge are incorporated is advocated (Fiore, 2012).

In addition, this report identifies the need to provide parents and educators with particular supports to ensure a mathematically-interactive and rich environment for children aged 3–8 years. It also indicates that the intensity of the support needs to vary according to the needs of particular groups of children (e.g., Ehrlich, Levine, & Goldin-Meadow, 2006).

Key Implications

The following are the key implications that arise from this report for the development of the mathematics curriculum for children aged 3–8 years:

- In the curriculum, a view of all children as having the capacity to engage with deep and challenging mathematical ideas and processes from birth should be presented. From this perspective, and in order to address on-going concerns about mathematics at school level, a curriculum for 3–8 year-old children is critical. This curriculum needs to take account of the different educational settings that children experience during these years.
- The curriculum should be developed on the basis of conversations amongst all educators, including those involved in the NCCA's consultative structures and processes, about the nature of mathematics and what it means for young children to engage in doing mathematics. These conversations should be informed by current research, as synthesised in this report and in Report No. 18, which presents a view of mathematics as a human activity that develops in response to everyday problems.
- The overall aim of the curriculum should be the development of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). As mathematization plays a central role in developing proficiency, the processes of mathematization should permeate all learning and teaching activities. These include connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. (*Chapter 1*)
- The curriculum should foreground mathematics learning and development as being dependent on children's active participation in social and cultural experiences, while also recognising the role of internal processes. This perspective on learning provides a powerful theoretical framework for mathematics education for young children. Such a framework requires careful explication in the curriculum and its implications for pedagogy should be clearly communicated. (*Chapter 2*)
- In line with the theoretical framework underpinning the curriculum, mathematical discourse (math talk) should be integral to the learning and teaching process. The curriculum should also promote the development of children's mathematical language in learning situations where mathematics development may not be the primary goal. Particular attention should be given to providing intensive language support, including mathematical language, to children at risk of mathematical difficulties. (*Chapter 3*)

- The goal statements of the curriculum should be aligned with its underlying theory. An approach whereby processes are foregrounded but content areas are also specified is consistent with a participatory approach to mathematics learning and development. In the curriculum, general goals need to be broken down for planning, teaching and assessment purposes. Critical ideas indicating the shifts in mathematical reasoning required for the development of key concepts should be identified. (*Chapter 4*)

- Based on the research which indicates that teachers' understanding of developmental progressions (learning paths) can help them with planning, educators should have access to information on general learning paths for the different domains. Any specification of learning paths should be consistent with sociocultural perspectives, which recognise the paths as provisional, non-linear, not age-related and strongly connected to children's engagement in mathematically-rich activity. Account needs to be taken of this in curriculum materials. Particular attention should be given to the provision of examples of practice, which can facilitate children's progression in mathematical thinking. (*Chapter 5*)

- The curriculum should foreground formative assessment as the main approach for assessing young children's mathematical learning, with particular emphasis on children's exercise of agency and their growing identities as mathematicians. Digital technologies offer particular potential in relation to these aspects of development. The appropriate use of screening/diagnostic tests should be emphasised as should the limitations of the use of standardised tests with young children. The curriculum should recognise the complex variety of language backgrounds of a significant minority of young children and should seek to maximise their meaningful participation in assessment. (*Chapter 6*)

- A key tenet of the curriculum should be the principle of 'mathematics for all'. Central to this is the vision of a multicultural curriculum which values the many ways in which children make sense of mathematics. While there are some groups or individuals who need particular supports in order to enhance their engagement with mathematics, in general distinct curricula should not be advocated. (*Chapter 7*)

- Curriculum developments of the nature described above are strongly contingent on concomitant developments in pre-service and in-service education for educators at preschool and primary levels.

A View of Mathematics

This report is concerned with definitions, theories, stages of developments and progression in mathematics in early childhood and primary education for children aged 3–8 years. It is premised on a view of mathematics as not only useful and as a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004).

Mathematics is intrinsic to our comprehension of the world. Stewart (1996) gives an overview of the many patterns that are found in nature and refers, in particular, to the pattern of number (e.g., the Fibonacci numbers and petals of flowers), the patterns of form (e.g., those found in sand dunes) and the pattern of movement (e.g., the regular rhythm of the human walk). He maintains that mathematics helps us to understand nature:

> *Each of nature's patterns is a puzzle, nearly always a deep one. Mathematics is brilliant at helping us to solve puzzles. It is a more or less systematic way of digging out the rules and structures that lie behind some observed pattern or regularity, and then using those rules and structures to explain what's going on. Indeed, mathematics has developed alongside our understanding of nature, each reinforcing the other.* (p. 16)

Appreciation of all of these facets of mathematics greatly enhances children's capacities to engage fully with the world around them.

Mathematics also has a utilitarian aspect. Struik (1987) describes how, as far back as the Old Stone Age, there was a need to measure length, volume and time. Nowadays, the availability of increasingly sophisticated tools allows ever-more accurate measurements of a myriad of attributes to be obtained. Wheeler and Wheeler (1979) suggest that mathematics is a language:

> *Mathematics is the language of those who wish to express ideas of shape, quantity, size and order. It is the language that is used to describe our growing understanding of the physical universe, to facilitate the transactions of the market place, and to analyze and understand the complexities of modern society. Thus, to communicate effectively, it is essential to have a knowledge of the language.* (p. 3)

Others talk about the beauty and joy of mathematics. For example, Poincaré's 'Aha' moment (the discovery of a new expression for Fuchsian functions) as he stepped on a bus is often cited to illustrate the stages of the creative process (e.g., Hadamard, 1945; Koestler, 1969). In interviews conducted with 70 mathematicians about their work, Burton (2004) found that the majority of her participants identified something which they termed intuition, insight, or, in a few cases, instinct as a key factor in coming to know mathematics – this insight was linked with a sense of joy. Dreyfus and Eisenberg (1986) suggest that just as individuals come to appreciate music, art and literature by understanding their underlying structures, so too they can appreciate mathematics.

However, mathematics is also linked with power. Since mathematics is behind most of society's inventions (not all for the common good!), it tends to give those who succeed in it access to wealth and power. It thus acts as a 'gatekeeper' – studies around the world show that gender, ethnicity and social class can impact on successful performance in mathematics and thus a large part of the world's population is denied access to its 'power' (e.g., Ernest, Greer, & Sriraman, 2009; Secada, 1995). While power and wealth may not seem to be of immediate concern to 3–8 year-old children, the foundations of mathematical proficiency are established during these years. Different conceptualisations of what it is to do mathematics can ameliorate such inequities and this is given attention throughout this report.

In the words of Hersh (1997, p. xi), 'mathematics is a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context'. Thus, this report and the accompanying one (Research Report No. 18) are also founded on a view of all individuals having an innate ability to solve problems and make sense of the world through mathematics.

Introduction

In this introduction we describe the broad context in which the development of a revised mathematics curriculum for children in the 3–8 year age range is embedded. This includes a description of current provision of early childhood education in Ireland. It also includes consideration of the existing Primary School Mathematics Curriculum (PSMC) and issues around its implementation, a review of performance on national and international assessments of mathematics, and an overview of recent policy initiatives related to mathematics education. Following this we look at the evolving language context in which mathematics education is provided in Irish schools and we acknowledge the range of social issues that can impact on children's mathematics learning in early education settings. We conclude with an overview of the remaining chapters in the report.

Context

The profile of mathematics as a curriculum area has increased greatly in recent years as countries seek to establish 'knowledge-based' or 'smart' economies, where many positions require a strong knowledge of mathematics, science or related areas (e.g., Commission of the European Communities, 2011). In educational circles, there is a concern to ensure that adequate numbers of students choose to study STEM subjects (science, technology, engineering and mathematics) at school, particularly at advanced levels (e.g., Jeffes et. al, 2012). In Ireland, a shift towards a knowledge-based economy has been signalled in government reports (e.g., Department of the Taoiseach, 2008) and policy documents (e.g., Department of the Taoiseach, 2011). These moves have been accompanied by a strong reform agenda in education, including the introduction of *Aistear*, a curriculum framework for children in preschool and in the early years of primary school (NCCA, 2009a), and revised syllabi in mathematics at post-primary level (The *Project Maths* initiative). Now the focus has shifted to mathematics at preschool and early primary levels.

Developing Mathematics Education in Ireland for Children Aged 3–8 Years

Preschool education and care in Ireland is to a large extent provided by community and voluntary agents and agencies, supported by grant aid from the government. In January 2010 a 'free preschool year' was introduced. The objective of this Early Childhood Care and Education Programme, which is open to both community and commercial service providers, is to benefit children in the key developmental period prior to starting school. Approximately 63,000 children participated in the preschool year in the first year of its implementation. The free preschool year is now available to all eligible children in the year before they attend primary school and there is the possibility that in the near future this will be extended to two years.

Children in Ireland can be enrolled in primary schools from the age of four, and up to recently half of all four-year-olds and almost all five-year-olds were enrolled in infant classes in primary schools. Also, there are approximately 1,600 three-year-old children, deemed to be at risk of educational disadvantage, enrolled in half-day preschool sessions in Early Start units in primary schools. The Delivering Equality of Opportunity in Schools (DEIS) programme[1] extends additional supports for schools in areas of economic and social disadvantage. The DES also provides various targeted supports for young children with special educational needs.

Aistear: the Early Childhood Curriculum Framework (National Council for Curriculum and Assessment (NCCA), 2009a) provides guidance and support for all adults working with the youngest children (birth to six). Sample learning opportunities related to the themes of *Communicating* and *Exploring and Thinking* illustrate in a general way how educators can support the development of various aspects of mathematical thinking and learning with toddlers and young children. However, because *Aistear* is a framework and not a curriculum, it does not provide specific guidance related to mathematics learning and teaching. The PSMC provides guidance for teachers of children from the age of 4 years. While children attending preschools may engage in many activities which promote mathematical learning and development, there is no systematic specification of these. Preschools may choose to structure their work within a particular curriculum such as *High Scope* or *Montessori*, they may use a variation on these, or they may develop their own curriculum.

Opportunities now exist for a systematic approach to rethinking mathematics education for all children aged 3–8 years. A revised approach should address the mathematical learning of children in preschool education, and also the dual, overlapping approaches described above in relation to official guidance across the age-range. It should be based on the understanding that mathematics learning begins early in the home and needs to be supported in a structured way right from the beginning of preschool education. It should also be predicated on findings that high quality early childhood education is a critical factor in ensuring that the mathematics potential of all children is realised and that existing equity gaps are closed (e.g., Bishop & Forganz, 2007; Ginsburg, Lee & Boyd, 2008; Perry & Dockett, 2008).

1 DEIS (Delivering Equality of Opportunity in Schools) is an action plan put in place by the (now) Department of Education and Skills in 2005 to address the effects of educational disadvantage in schools. The School Support Programme (SSP) under DEIS comprises a set of measures that provides schools with additional human and material resources to tackle educational disadvantage, in schools with the highest levels of assessed disadvantage. Urban schools in the SSP are allocated to Band 1 or Band 2, depending on their level of disadvantage. There is a separate set of measures for rural schools.

Curriculum Context

The current Primary School Curriculum (PSC) (Government of Ireland, 1999) was introduced in 1999, with in-service for mathematics provided in 2001–02, and implementation beginning in 2002–03 (DES, 2005). While maintaining some important links with its predecessor, *Curaclam na Bunscoile* (DE, 1971), the PSMC also drew heavily on Vygotskian ideas about teaching and learning, in that it emphasised the social aspects of mathematics development, the importance of language in acquiring mathematical knowledge, and the key role of the teacher in modelling and supporting children's emerging understanding of mathematics.

The PSMC, which is based on socio-constructivist and guided-discovery theories of learning, aimed to equip children with a positive attitude towards mathematics, to develop problem-solving abilities and the ability to apply mathematics to everyday life, to enable children to use mathematical language effectively and accurately, and to enable them to acquire an understanding of mathematical concepts and processes, as well as proficiency in fundamental skills and basic number facts.

The PSMC was generally well-received by teachers. In a review of curriculum implementation by the NCCA (2005), a majority of teachers reported an increased emphasis on practical work as its greatest success, while enjoyment of mathematics by children was also highlighted. The implementation of practical activities on a daily basis, especially for Measures[2], was also noted. About half of teachers reported that catering for the range of children's mathematical abilities represented their greatest challenge, with inadequate instructional time contributing to this. Significantly, teachers of junior and senior infants identified classifying, matching and ordering as the strand units that had most impact on their planning and teaching[3]. Data was identified as the strand that teachers struggled with most often.

An evaluation of curriculum implementation by the Inspectorate of the then Department of Education and Science (DES, 2005), which was mainly based on observations of the work of teachers in teaching mathematics in school settings, found that the PSMC was not being implemented successfully in a significant minority of schools and classrooms. For example, some difficulties with implementation were noted, especially for Shape and Space, where children in one-third of observed classes were able to name shapes, but were not familiar with their properties, and for Data, where scope was identified for the development of specific skills and the use of integration, linkage, and a stronger cross-curricular approach. In the case of teaching

2 Measures is one of five strands in the curriculum for all grade bands. The others were Number, Shape & Space, Algebra and Data.

3 Whereas the PSMC has five strands at all grade bands, an additional strand – Early Mathematical Activities – is included in the curriculum for junior and senior infant classes, and its strand units are Classifying, Matching, Comparing and Ordering.

Problem-Solving[4], where weaknesses were also apparent in one-third of classrooms, inspectors referred to non-implementation of the school plan with respect to problem-solving, and 'an over-reliance on traditional textbook problems, which did not promote the development of specific problem-solving skills' (p. 29). In considering the use of guided discovery methods and concrete materials, inspectors noted that learning in one-third of classes 'was passive and reliant on activities that lacked focus and required more purposeful direction by the teacher' (p. 29). Inspectors also noted an 'over-reliance on whole-class teaching, where teacher talk dominated and where pupils worked silently on individual tasks for excessive periods' (p. 30). A number of difficulties with the assessment of mathematics were also noted, including inappropriate use of standardised tests and an absence in some classrooms of continuous records of children's achievement.

Performance Context

A number of studies, both national and international, have raised concerns about performance among children in Ireland on specific aspects of mathematics, and, in some cases, on overall mathematical performance.

The 2009 National Assessment of Mathematics Achievement (Eivers et al., 2010) – the first national assessment since the introduction of the PSMC to assess mathematics in both second and sixth classes – reports poorer performance on items designed to assess Measures at both class levels, compared with other content strands, and a decline in performance on Measures and Shape and Space between second and sixth; performance on items designed to assess the Applying and Problem-Solving process skill was weak at both second and sixth classes (a finding that also emerged in earlier national assessments conducted at other grade levels). Other problematic areas were integrating mathematics into other subject areas (61%), working with lower-achieving children in mathematics (60%), and extending the abilities of higher-achieving children (56%).

In the mathematics component of the Trends in International Mathematics and Science Study (TIMSS), administered to children in fourth class in over 50 countries in 2011, Ireland achieved a mean score (527) that was significantly above the international average, but significantly below the mean scores of 13 countries/regions, including Northern Ireland (562), Belgium (Fl.) (549), Finland (545), England (542), the United States (541), the Netherlands (540) and Denmark (537), as well as several Asian countries. Further, whereas 9% of children in Ireland achieved at the Advanced International Benchmark (the highest 'proficiency' level on TIMSS maths), 43% of children in Singapore, 39% in Korea and 24% in Northern Ireland did so (Eivers & Clerkin, 2012). Relative to their performance on the test as a whole, Irish children performed quite well on the TIMSS content area of Number, and less well on Geometric Shapes & Measures

4 Applying and Problem-Solving is one of six process skills in the PSMC which are taught at all grade bands. The others are Communicating and Expressing, Integrating and Connecting, Reasoning, Implementing and Understanding and Recalling.

and on Data. On the process subscales, children in Ireland performed relatively well on Knowing items, and quite poorly on Reasoning items, including items requiring problem-solving abilities.

The relatively disappointing performance of children in Ireland on TIMSS mathematics contrasts with the performance of the same children on a related study administered at the same time – the 2011 Progress in International Reading Literacy Study (PIRLS). Just five countries had mean scores that were significantly higher than Ireland's in PIRLS and the proportion of high achievers in Ireland was about the same as in other high-scoring countries (Eivers & Clerkin, 2012).

In 2009, 15-year-olds in Ireland performed significantly below the average for OECD countries on the mathematics component of the Programme for International Student Assessment (PISA), ranking 26th of 34 OECD member countries. Further, 21% of students in Ireland performed at or below Level 1 on the PISA mathematics scale. This is interpreted by the OECD (2010) as indicating that they lack the mathematics skills needed for everyday living and/or future study. While the size of the decline in performance on PISA mathematics in Ireland between 2003 (503 points) and 2009 (487) has been disputed (Perkins et al., 2012), the relatively disappointing performance by children in Ireland in mathematics across PISA cycles is worth noting, in a context in which performance on reading literacy (except in 2009) and scientific literacy have been above their respective OECD averages. Concern must also be expressed at the relatively poor performance of students in Ireland on the Space and Shape component of PISA mathematics in 2003 and 2012, when their mean scores were significantly below the corresponding OECD average. PISA Shape and Space items require students to solve problems that include shapes in different representations and dimensions (Cosgrove et al., 2005; Perkins et al., 2013). Female students in Ireland performed particularly poorly on this PISA content domain.

Notwithstanding differences between national curricula/syllabi and the assessment frameworks accompanying international studies (e.g., Close, 2006), the relatively disappointing overall performance of children in Ireland on international studies of mathematics achievement is a matter of concern, given current concerns about standards in mathematics, the role of mathematics in other subjects, and efforts to encourage students to select STEM subjects, especially at upper-secondary level. Related to this, it is a matter of concern that problem-solving presents a significant difficulty for children in Ireland from at least second class onwards. Without a strong foundation in this important process, many children may not reach their potential in mathematics.

Policy Context

There have been two significant policy initiatives in mathematics education in recent years. The first, *Project Maths* (PM), a new initiative to change the teaching and assessment of mathematics in post-primary schools, has been underway on a phased basis since 2008, and many aspects of the revised PM syllabi have now been implemented in all schools. The broad aims of PM, which is based on sociocultural theories of mathematics, are to equip students at Junior and Leaving Certificate levels with:

- the ability to recall relevant mathematical facts
- instrumental understanding ('knowing how')
- relational understanding ('knowing why')
- the ability to apply their mathematical knowledge and skill to solve problems in familiar and unfamiliar contexts
- analytical and creative powers in mathematics
- an appreciation of mathematics and its uses
- a positive disposition towards mathematics (Government of Ireland, 2012, p. 6).

Important features of *Project Maths* include the following:

- an acknowledgement of the continuum of mathematics development that extends from early childhood through post-primary schooling, with an emphasis on connected and integrated mathematical understanding
- efforts to establish links between mathematics learning at primary level through the implementation of a common introductory course in the first year of post-primary schooling
- establishment of a learning environment for problem-solving, in which problem-solving permeates all aspects of mathematics learning, and students consolidate previous learning, extend their knowledge, and engage in new learning experiences
- engagement with a wide variety of mathematical problems, some of which are purely mathematical, and others more applied
- links within strands of study to other subjects
- a focus on conceptual understanding (Government of Ireland, 2012, p. 8).

The effects of PM are as yet unclear. An initial study (Jeffes et al., 2012) was somewhat positive about the performance of samples of Junior Certificate and Leaving Certificate students[5] on tests of mathematics administered in spring 2012 that were benchmarked against international standards. However, no significant differences in performance were found between students in initial PM schools (where implementation of PM began in 24 schools in 2008) and other schools (where implementation began in 2010). Nevertheless, students in the initial schools were found to engage more often in the types of activities associated with PM (applying mathematics to real-life

5 Junior Certificate students in other schools had not studied any of the new syllabus materials at the time of testing; Leaving Certificate students in other schools had studied some aspects.

situations, conducting investigations, and participating in discursive and collaborative activities), compared with students in schools where PM had not been fully implemented. A follow-up report on the implementation of PM (Jeffes et al., 2013) again raised issues about the extent to which PM was being implemented effectively in schools. A review of student materials found evidence of a strong emphasis on implementing mathematical procedures and, to a lesser extent, problem solving, but 'little evidence that students are demonstrating reasoning and proof and communication, or making connections between mathematics topics' (p. 5).

The relevance of PM to mathematics in early years and primary school settings concerns the extent to which proposals for change among children in the 3–8 years range might need to be broadly consistent with the goals and methodologies underpinning *Project Maths*, including a substantially-increased emphasis on problem-solving, and a strong focus on the application of mathematical ideas in real-life contexts. In considering this it can be noted that these two elements are key features of the current PSMC.

In the second policy initiative, the Department of Education and Skills published a *National Strategy to Improve Literacy and Numeracy Among Young People 2011–20* (DES, 2011). The strategy made a strong case for improving standards in literacy and numeracy across all levels of the education system, and set out a series of actions designed to bring about improvement, including

- an increased focus on literacy and numeracy across the curriculum, including increased allocation of time to the teaching of English and mathematics, some of which could involve cross-curricular activities
- the clear specification of learning outcomes in revised curricula and the provision of exemplars to illustrate such outcomes
- the extension of the *Aistear* early childhood framework (NCCA, 2009a; 2009b) to children in the 4–6 years age range (i.e., those in the infant classes in primary school)
- the achievement of specified targets in the *National Assessment of Mathematics* at second and sixth class (an increase in the proportion of children achieving at the highest proficiency levels, and a reduction in the proportion achieving at the lowest levels)
- the achievement of an increase in the percentage of students taking the Higher Level mathematics examination in Leaving Certificate to 30% by 2020 (it was 22% in 2012).

These actions, together with a range of related measures in the areas of teacher education and teacher professional development, are intended to result in a significantly changed educational landscape over the next few years, compared with that in place when the PSC was introduced in 1999.

Linguistic and Social Contexts

Significant demographic changes have occurred in Ireland since the PSC was introduced in 1999, including greatly increased participation of children in the education system who do not speak the language of instruction (English or Gaeilge) at home. In the 2009 *National Assessment of Mathematics*, 15% of children in second class who were born outside Ireland had a mean score that was lower than that of Irish-born children, but the difference was not statistically significant (Eivers et al., 2010). However, 10% of children (mainly born outside of Ireland) reported speaking a language other than English or Irish most often at home, and these children had a significantly lower mean score (by 22 points) than speakers of English. Interestingly, the difference between the latter groups at sixth class was 12 points, and it was not statistically significant. These outcomes point to challenges faced by children who speak a language other than English at home in learning mathematics. They also point to the need to develop language in the context of teaching mathematics, and suggest that progress can be made over time.

Another linguistic context is that in which children learn mathematics through the medium of Irish. In a study of mathematics performance in Irish medium schools in 2010, children in second class in Gaelscoileanna achieved a mean score that was significantly higher, by one-sixth of a standard deviation, than the average score obtained by a national sample in the 2009 National Assessment (Gilleece Shiel, Clerkin, & Millar, 2012). However, by sixth class, children in Gaelscoileanna had a mean score that was not significantly different from the national average.[6] The latter result is particularly interesting as the same children achieved a mean score in English reading that was one-third of a standard deviation above the national average – with the strong performance in English reading attributed to higher socio-economic status among children in Gaelscoileanna.

Gilleece et al. also found that children in Gaeltacht schools in which Irish was the medium of instruction achieved a mean score that was not significantly different from the national average in second class, and was significantly higher in sixth class.

The outcomes of this study point to the challenges faced in teaching mathematics in Irish-medium contexts, and to issues around assessment of mathematics, including the language of assessment (i.e., whether children are assessed in Irish, English, or a combination of the two languages) and instruction (whether, to what extent, and how English is used in mathematics classes).

Socio-economic status has been identified as a factor associated with mathematics achievement. In the 2009 National Assessment of Mathematics Achievement, children in second class attending DEIS Band 1 urban schools (those with the highest levels of socio-economic disadvantage) achieved a

6 In second class, 91% of pupils in Gaelscoileanna and 49% in Gaeltacht schools completed the test in Irish. The corresponding figures for sixth class were 81% and 59% respectively. It is unclear whether all pupils in sixth class taking the test through Irish were able to demonstrate the full range of their mathematical competencies.

mean score (217) that was lower than the mean score of children in DEIS Band 2 schools (228), and significantly lower than children in non-DEIS urban schools (253) (Eivers et al., 2010). Children attending DEIS rural schools (266) and children attending non-DEIS rural schools (259) also achieved scores that were significantly higher than children attending DEIS Band 1 schools. Outcomes were broadly similar at sixth class, where there was also a difference of 40 points (four-fifths of a standard deviation) between children in DEIS Band 1 urban schools and those in non-DEIS urban schools.

There is evidence that some of the differences in mathematical achievement found in school settings may have their origins in children's home backgrounds. In TIMSS 2011, fourth class children in Ireland and on average across participating countries who had 'some' or 'few' human resources at home[7] achieved a mean mathematics score that was significantly lower (by one-half of a standard deviation) than that of children with 'many' resources (Mullis, Martin, Foy, & Arora, 2012). The relationship between home environment and mathematics achievement may be mediated by the types of language and mathematical activities – whether formal or informal – in which children engage in their home. International research (e.g., Sylva, Melhuish, Sammons, Siraj-Blatchford & Taggart, 2004) indicates that it is what parents do with children at home, rather than who they are, that is of most significance to children's early learning.

International research has identified gaps in children's mathematical knowledge well before they start school, in particular among children living in disadvantaged circumstances (e.g., Jordan & Levine, 2009), with more marked differences on tasks requiring language (Jordan et al., 2006). This issue is discussed further in *Chapter 3*.

Overview of Chapters

Educators' beliefs are strongly associated with how they see mathematics and mathematics education. Thus, the opening chapter of this report presents three conceptions of mathematics and their different implications for mathematics education. It emphasises current views of mathematics education as a cultural phenomenon, where issues of equity and access are paramount and where numeracy is seen as one aspect of mathematics. The concept of mathematical proficiency is presented as an overarching aim, with mathematization as integral to its achievement.

While a wide range of theories are available for explaining mathematical learning and development during early childhood, in *Chapter 2* we focus on the perspectives afforded by cognitive and sociocultural theories. These perspectives are the ones that underpin key developments in mathematics education over the last decade. Constructionism is also highlighted because of its importance in underpinning recent developments in digital learning and in the use of digital tools for learning.

7 Level of human resources at home was based on an index that included number of children's books in the home, at least two home study supports (internet connection, own room), parental occupations and parental education.

In Chapter 3, the role of language and communication in young children's mathematical development is considered. Ideas about developing children's mathematical vocabulary and their engagement in math talk are elaborated on. The mathematical language needs of children in disadvantaged circumstances and those with language impairment are also considered.

Two concepts arise as we explore the task of identifying goals for early childhood mathematics education in Chapter 4: the concept of 'big ideas'; and that of 'powerful mathematical ideas'. Differences in emphases between the two approaches to the specification of curriculum goals are discussed. These are compared with the approach used to specify content and skills in the 1999 PSMC. The level of detail that might be employed in the specification of goals is also an issue addressed in this chapter.

Chapter 5 traces ideas about stages of development from those associated with Piaget to current conceptions of learning trajectories or learning paths. The literature shows how a cognitive perspective may give rise to interpretations of children's thinking about mathematical concepts as predetermined. This is contrasted with a sociocultural/situative stance where changes in levels of understanding are explored in order to clarify the particular paths that children take. We discuss the implications of these different perspectives for learning and teaching.

The range of methods for the formative assessment of children's mathematical learning is reviewed in Chapter 6, including observations, tasks, interviews and conversations. Consideration is also given to the appropriateness of using more formal assessments, including screening/diagnostic assessments. Potential difficulties relating to the use of standardised tests with children in the age range of 3–8 years are highlighted. This chapter also considers assessment of children with special educational needs and those for whom their first language is not the language used in the education setting.

Chapter 7 focuses on how preschools and schools might address equity issues in learning mathematics. The perspective we present is that groups of children identified as at-risk of underachieving because of a learning disability or talent do not require distinctive teaching approaches, but that account needs to be taken of their individual learning needs. We also identify other groups who may appear to be underachieving because of cultural/social factors, and suggest how provision can be made to address their particular needs.

Chapter 8 outlines the key implications for the redevelopment of the PSMC for children from 3–8 years of age arising from the report.

CHAPTER 1

Defining Mathematics Education

Mathematics learning begins from birth as children explore the world around them. As they develop, they are supported in their learning by the people around them. The environment is a rich resource for engaging with mathematics, especially when it provides opportunities to listen to and use mathematical language and to engage in mathematical ways with everyday experiences. Through the assistance of others, children's attention and activity are directed in ways that enable them to reason and to grow in their abilities to communicate mathematically. As they do so, they develop an affinity with mathematical tools and they take pleasure and interest in thinking mathematically.

In order to facilitate children's journeys into the world of mathematics and to afford them a rich experience of the subject, it is important to give consideration to issues related to the foundations of mathematics, what it means to engage in mathematics and the key aims and goals for mathematics education at preschool and primary levels. These matters are explored further in this chapter.

The Foundations of Mathematics

Davis and Hersh (1981) suggest that three standard dogmas are usually presented in discussions on the foundations of mathematics – Platonism, formalism and constructivism. Platonists are of the view that mathematical objects (e.g., geometric shapes) are real and objective and that their existence is independent of an individual's knowledge of them. Those who adopt the formalist perspective believe that there are no mathematical objects and that mathematics comprises definitions, theorems and axioms. What matters to them are the rules and how one formula can be transformed into another. According to constructivists, mathematics is comprised only of those objects that individuals construct themselves. Those who hold this conception of mathematical knowledge view it as 'tentative, intuitive, subjective and dynamic' (Nyaumwe, 2004, p. 21). Hersh (1997) argues that each of these three views is limited, e.g., Platonism denies the human dimension of mathematics while constructivism fails to explain the universality of mathematical knowledge (see also Stemhagen, 2009) and, therefore, he adopts a humanist stance, that is:

- Mathematics is human – it is part of and fits into human culture.
- Mathematical knowledge is not infallible. Like science, mathematics can be advanced by making mistakes, correcting and re-correcting them.

- There are different versions of proof and of rigour depending on time, place and other things. For example, the use of computers in proof is a recent phenomenon.
- Mathematical objects are a distinct variety of social-historic objects. Like literature or religion, they are a special part of culture. (ibid., p. 22)

From this perspective, mathematical entities derive from the needs of everyday life (e.g., in science or technology) and have no meaning beyond that ascribed to them by a shared human consciousness.

Since the 1980s, there has been considerable interest in the relationship between teachers' conceptions of mathematics and their pedagogical practices (e.g., Dossey, 1992; Ernest, 1989; Leder, Pehkonen, & Törner, 2003). Ernest (1989) argues that a teacher's conceptions or set of beliefs about the nature of mathematics as a whole forms the basis of his/her philosophy of mathematics. However, these conceptions are not necessarily consciously held views or fully articulated philosophies. Drawing on the work of Thompson (1984), Ernest (1989) identified three conceptions of mathematics held by teachers:

1. Platonist: view of mathematics as a static but unified knowledge that can be discovered rather than created.
2. Problem-solving: view of mathematics as on-going enquiry and coming to know.
3. Instrumentalist: view of mathematics as a 'bag of tools' made up of utilitarian facts, rules and skills.

These conceptions of mathematics align with philosophies of mathematics as described above (Dossey, 1992; Ernest, 1989). They have different implications for mathematics education. Most notably, the view of mathematics as 'absolute and certain' is often perceived as eliminating learners, particularly women and marginalised groups, from the subject – '[n]ot only is the personness of the discipline removed, but hierarchy of knowledge and elitism of knowers construes an antagonistic cultural climate in classrooms' (Burton, 2001, p. 596). On the other hand, a view of mathematics as co-constructed promotes student engagement and critical thinking (e.g., Povey, 2002; Stemhagen, 2009). Since a teacher's beliefs about mathematical content, the nature of mathematics and its teaching and learning are strongly associated with what he or she does in the classroom (e.g., Törner, Rolka, Rösken, & Sriraman, 2010), any proposed change in the curriculum rests on addressing these beliefs.

A Definition of Mathematics Education

According to Valero (2009), mathematics education can refer to two different domains: a field of practice where people engage in the activities connected to the teaching and learning of mathematics and a field of study which is the space of scientific enquiry on and theorisation about the field of practice. It is his contention that the field of study defines the field of practice and since the former is often focused on the relationship between teacher, learner and mathematical content, broader social, cultural and political factors are overlooked. He argues for broadening the scope of mathematics education:

> *Let us think about mathematics education as a field of practice covering the network of social practices carried out by different social actors and institutions located in different spheres and levels, which constitute and shape the way mathematics is taught and learned in society, schools and classrooms…This broader definition of the field evidences the social, political, cultural and economic dimensions that are a constitutive element of mathematics education practices.* (p. 240)

Current views of mathematics education are inextricably linked with ideas about equity and access (Bishop & Forganz, 2007) and reflect this broader scope. In attempting to define mathematics education, one is forced to consider questions such as 'What mathematics?' and 'Mathematics for whom?' and 'Mathematics for what purpose?'. Bishop (1988, as cited in Bishop & Forganz, 2007), in seeking to answer the first of these questions, differentiates between Mathematics with an upper-case M and mathematics with a lower-case m, both of which, in his view, should be addressed by schools. He regards the former as the universal mathematics that is the basis of mathematics curricula in schools, while the latter refer to a wider mathematical knowledge that is used in everyday life in a particular society or culture. Current views of mathematics education assume that we are talking about mathematics education for all children. In clarifying the meaning of 'mathematics for all', Clements, Keitel, Bishop, Kilpatrick & Leong (2013) suggest that it is 'the kind of goal that anticipates a world in which all people have the opportunity to learn, and benefit from learning, mathematics' (p. 8). In response to the third question, the purposes of learning mathematics in schools are now seen as twofold: the preparation of mathematically-functioning citizens of a society and the preparation of some for future careers in which mathematics is fundamental. Bishop and Forganz (2007) state that, from an equity perspective, no one should be denied access to participation along this path. Clements et al. (2013) draw attention to the fact that, in learning mathematics, conditions and context are crucial. They, in common with others (e.g., van den Heuvel-Panhuizen, 2003), reiterate the fact that mathematics is a cultural phenomenon and that the forms of mathematics in schools should 'arise out of, and are obviously related to, the needs of learners, and the societies in which they live' (Sriraman & English, 2010, p. 33).

Numeracy

Numeracy is a term having a range of different definitions, many of which encompass the equity and access ideals of 'mathematics for all' and ideas related to competent citizenship (Bishop & Forganz, 2007). Clements et al. (2013) suggest that the concept of numeracy, while still ill-defined, has gradually been extended over the years 'beyond purely arithmetical skills to embrace not only other elementary mathematical skills but also affective characteristics such as attitudes and confidence' (p. 32).

Bishop and Forganz (2007) suggest a number of possible relationships between mathematics and numeracy:

1. Mathematics and numeracy intersect, that is they share aspects but do not include each other.
2. Numeracy is a subset of mathematics.
3. Mathematics is a subset of numeracy.
4. Numeracy is mathematics.
5. Mathematics and numeracy are two very different phenomena, having no relationship.

The term numeracy has been used in recent years by a number of governments, including those of Canada, Australia and Ireland, to describe aspirations for aspects of mathematics learning including quantitative literacy. Numeracy, as it is generally envisaged in such statements, is seen as something which is not limited to the ability to use numbers, but, for instance, as 'the capacity, confidence and disposition to use mathematics to meet the demands of learning, school, home, work, community and civic life' (DES, 2011, p. 9). This is very similar to that used by the Australian Association of Mathematics Teachers (AAMT, 1998) which states that 'To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life' (p. 2).

The statement from AAMT goes on to describe the place of numeracy in the curriculum:

> *In school education, numeracy is a fundamental component of learning, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:*
>
> - *underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic);*
> - *mathematical thinking and strategies;*
> - *general thinking skills; and*
> - *grounded appreciation of context.* (p. 2)

This suggests a view of numeracy as involving the use of mathematics, but not as the same as mathematics. The DES definition above is in the same vein. It seems then that both the DES definition and that from AAMT correspond with a view of numeracy as a subset of mathematics, Option 2 in the Bishop and Forganz list above. We conclude then that the definitions offered above are not talking about mathematics per se but rather about a subset of mathematics which is developed in school education. While the development of numeracy is important, education at all levels should encompass a broader view of mathematics.

Defining Mathematics Education for Children Aged 3–8 Years

Aistear defines numeracy as 'developing an understanding of numbers and mathematical concepts' (2009a, p. 56). It views mathematical literacy, whereby children learn to communicate using the mathematics sign system, as part of being literate. Perry and Dockett (2008) argue that numeracy, mathematical literacy and mathematics go hand-in-hand in early education settings because young children's learning takes place in the context of holistic learning experiences and in contexts that are part of their day-to-day lives:

> *The contextual learning and integrated curriculum apparent in many early childhood – particularly prior-to-school settings – ensures that there is little distinction to be drawn between numeracy, mathematical literacy and aspects of mathematical connections with the children's real worlds.* (p. 83)

Concepts of number and operations with numbers are identified as being at the heart of mathematics for young children (NRC, 2001). But prior to children developing concepts about number, mathematical thinking begins for all children with comparisons of quantity and the development of an understanding of quantity (e.g., Griffin, 2005; Sophian, 2008). This does not mean though, that curricula for early childhood should be limited to the topic of number. Rather, as children are gradually introduced to mathematics in early education settings, it should address the range of mathematical ideas that all children need to engage with in order to reach their potential in their mathematics learning. It should also encompass all of the topics of shape and space/geometry and measure, data, and algebra (e.g., Saracho & Spodek, 2008; Ginsburg, 2009a; Clements & Sarama, 2004).

There are now a number of sources that educators can look to for advice on what principles should guide mathematics education for young children. These include statements from The National Association of Educators of Young Children (NAEYC) in the United States who joined forces with the National Council of Teachers of Mathematics (NCTM) to issue a position paper (2002/2010) on early childhood mathematics. Similarly, in Australia Early Childhood Australia and the Australian Association of Mathematics Teachers set out their position on what mathematics education for young children should be (AAMT/ECA, 2006). General principles which should underpin pedagogy/practice are explored in Report No. 18, Chapter 1, Sections: *Principles that Emphasise People, Relationships and the Learning Environment* and *Principles that Emphasise Learning*.

A Key Aim of Mathematics Education: Mathematical Proficiency

Mathematical proficiency has been adopted as a key aim in policy documents on mathematics in many countries, for example, the US (CCSSM/NGA, 2010), New Zealand (Anthony & Walshaw, 2007) and Australia (National Curriculum Board, 2009). Mathematical proficiency comprises the following five interwoven strands (NRC, 2001, pp. 116–133):

- conceptual understanding – comprehension of mathematical concepts, operations, and relations

Individuals who have a conceptual understanding of mathematics know more than isolated facts. They have an integrated grasp of mathematical ideas and know why and in what context the ideas are applicable. They make connections between ideas, thus allowing them to retain facts and procedures.

- procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

Individuals who are procedurally fluent in the domain of number are able to analyse similarities and differences between methods of calculating. These methods include written procedures, mental methods and methods that use concrete materials and technological tools.

- strategic competence – ability to formulate, represent, and solve mathematical problems

Individuals who are strategically competent have the capacity to form mental representations of both routine and non-routine problems, and detect mathematical relationships, and are flexible in their problem-solving approaches. Strategic competence depends upon and nurtures both conceptual understanding and procedural fluency.

- adaptive reasoning – capacity for logical thought, reflection, explanation, and justification

A hallmark of adaptive reasoning is the justification of one's work. This justification can be both formal and informal. Individuals clarify their reasoning by talking about concepts and procedures and giving good reasons for the strategies that they are employing. Such justification is supported by collaboration with others and by the use of physical and mental representations of problems.

- productive disposition – habitual inclination to see mathematics as sensible, useful, worthwhile, coupled with a belief in diligence and one's own efficacy.

Individuals who have a productive disposition believe that mathematics is useful and relevant. They do not regard mathematics as being for the 'elite few' but rather as a subject in which all can enjoy success if they make appropriate effort.

Key to the development of mathematical proficiency is the interdependence and interconnection among the strands, demonstrated in Figure 1.1.

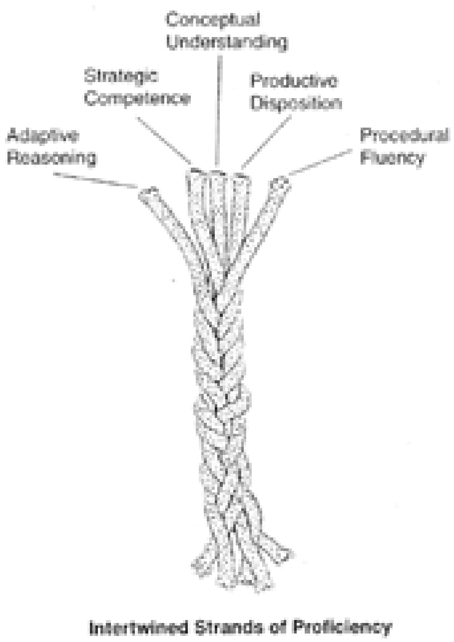

Figure 1.1: Intertwined Strands of Proficiency. From *Adding It Up: Helping Children to Learn Mathematics.* National Research Council (NRC) (2001, p. 5). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.

Given the breadth and depth of the concept of mathematical proficiency, we support its inclusion as a key aim for the revised curriculum. As we understand it, individuals become mathematically proficient over their years in educational settings. Each of the strands becomes progressively more developed as children's mathematical experiences become increasingly sophisticated.

As described above, mathematical proficiency is developed through engagement with processes such as communicating, reasoning, argumentation, justification, generalisation, representing, problem-solving, connecting and communicating. All of these are encompassed in the overarching concept of mathematization (Bonotto, 2005; NRC, 2009). Below, we introduce some ways in which the concept of mathematization is defined in the literature.

Mathematization

The Realistic Mathematics Education (RME) movement is illustrative of how a particular perspective on mathematics suggests a particular way of conceptualising mathematics education. Freudenthal (1973) thought of mathematics not as a body of knowledge that had to be transmitted but as a form of human activity. For him, the learning of mathematics meant involving children in 'mathematization' where, with appropriate guidance, they would have the opportunity to reinvent mathematics. Central to his learning theory was the notion of level-raising where what might be known informally at one level becomes the object of scrutiny at the next level. Treffers (1987) expands on level-raising by formulating the ideas of 'horizontal' and 'vertical' mathematization. In horizontal mathematization, the learner develops mathematical tools or symbols that can help to solve problems situated in real-life contexts. In vertical mathematization, the learner makes connections between mathematical concepts and strategies, that is, she or he moves within the world of mathematical symbols.

In the United States, the NRC report (2009) addressed the connection between mathematizing and mathematical processes:

> *Together, the general mathematical processes of reasoning, representing, problem solving, connecting, and communicating are mechanisms by which children can go back and forth between abstract mathematics and real situations in the world around them. In other words, they are a means of both making sense of abstract mathematics and for formulating real situations in mathematical terms – that is, for mathematizing the situations they encounter.* (p. 43)
>
> *Mathematizing happens when children can create a model of the situation by using mathematical objects (such as numbers or shapes, mathematical actions (such as counting or transforming shapes), and their structural relationships to solve problems about the situation. For example, children can use blocks to build a model of a castle tower, positioning the blocks to fit with a description or relationships among features of the tower, such as a front door on the first floor, a large room on the second floor, and a lookout tower on top of the roof. Mathematizing often involves representing relationships in a situation so that the relationships can be quantified.* (p. 44)

Ginsburg (2009a) argues that early childhood mathematics education should focus on mathematization. In his view, the educator's role is to support children in their efforts to mathematize. This involves 'helping them to interpret their experiences in explicitly mathematical form and understand the relations between the two' (p. 415). Often this support is offered in the course of everyday activities. The process of mathematization is also emphasised by others as a key aspect of early mathematics education (e.g., Perry & Dockett, 2008).

It follows then that mathematization fosters mathematical proficiency and so should be a key focus of early mathematics education.

Conclusion

High-quality mathematics education for children aged 3–8 years is predicated on opportunities for rich engaging interactions with knowledgeable educators who challenge children to think and communicate mathematically. They offer support for children's mathematizing, for their constructions of a good number sense and for their developing understandings of critical mathematical ideas. Educators use their knowledge of mathematics, of children's learning and of mathematics pedagogy to introduce children gradually to a structured curriculum which emphasises the development of mathematical proficiency. However, the implementation of such a curriculum is strongly linked not only to a teacher's beliefs about and attitudes towards mathematics, but also to those of the broader social arena. In particular, a view that mathematics is a human endeavour, deriving from the needs of everyday life, underpins the notion of 'mathematics for all'. Changes in the mathematics curriculum, therefore, depend on stakeholders in education engaging in conversations about mathematics education and its key aims and goals.

The key messages arising from this chapter are as follows:

- Mathematics is no longer considered to be a fixed, objective body of knowledge. Rather it comprises a number of social practices that are negotiated by the learner and teacher within the broader social, political and cultural arena. A teacher's conceptualisation of mathematics and what it is to do mathematics have strong influences on pedagogic practices.
- Current views of mathematics education are inextricably linked with ideas about equity and access. While the development of numeracy is important, a broad interpretation of mathematics should underpin efforts towards curricular reform in Ireland. A broad perspective is coherent with a view of mathematics as a human, socially-constructed and creative endeavour.
- Given the breadth and depth of the concept of mathematical proficiency, we propose that it be adopted as a key aim of the mathematics curriculum. It is promoted through engagement with processes such as connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. All of these are encompassed in the overarching concept of mathematization. Thus mathematization should be a key focus of mathematics education.

CHAPTER 2

Theoretical Perspectives

For many years psychological perspectives dominated conceptions of how children learn mathematics. A major development in the 1990s occurred with the social turn in mathematics education research (Lerman, 2000). This resulted in the increasing use of sociocultural theories to explain mathematical learning and development and a move away from seeing learning as acquisition of knowledge, to seeing learning as the understanding of practice (in this case, the practice of doing mathematics). In addition, a number of new perspectives have become visible recently, including social justice theory, networking theories and semiotics to name but a few. The emergence of all of these new ways of thinking about mathematics learning and the factors that influence it means that it is increasingly challenging to explain mathematics learning by reference to a narrow range of theories. Consequently, from the point of view of mathematics educators, a wide range of theories serve to explain children's mathematical learning and development and to influence mathematics education (e.g., Sriraman & Nardi, 2013).

Learning in early childhood, as at any age or stage of life, is generally considered to be a complex process not easily explained by a single theory or perspective (e.g., Dunphy, 2012). Within the field of early childhood education, social constructivist perspectives take account of the central role of social interaction in shaping learning. Sociocultural theories of learning, in addition to the social aspect, also consider culture and cultural influences as centrally important to learning. Cognitive perspectives arising from, for example, constructivist theories, are also useful because they emphasise the active, constructivist nature of human learning and development and the idea that we each construct our own learning.

In *International Trends in Post-Primary Mathematics Education* for the NCCA, Conway and Sloane (2005) identified three main theoretical perspectives on learning that have had a significant impact on mathematics education over the past hundred years. These included behavioural, cognitive and sociocultural theories. Behaviourist theories (which emphasise behaviour modification via stimulus response and selective reinforcement), while still influential in certain teaching practices, are no longer influential in mainstream mathematics research.

In this chapter we propose to build on Conway and Sloane's work by discussing sociocultural and cognitive perspectives as they pertain to young children's mathematical learning and development. These perspectives are central since they are the main perspectives underpinning recent significant research and developments in early childhood mathematics. The developments we refer to are discussed in both this report and Report No. 18. They include the current attention being given to curriculum goals (see *Chapter 4* in this report; Report No. 18, Chapter 3, Section: *Curriculum Goals*), learning and teaching paths in early childhood mathematics (see *Chapter 5* in this report; Report No. 18, Chapter 3, Section: *Content Areas*), as well as developments in pedagogy and assessment (see *Chapter 6* in this report; Report No. 18, Chapter 2, Section: *Meta-Practices*). These developments and their implications for the revision of the mathematics curriculum are discussed in later sections of this report. We also draw attention to the theory of constructionism, a theory of learning which takes cognisance of the role of cultural tools while also being consistent with cognitive and sociocultural theories. Constructionism's importance in this report is that it underpins the discussion in Report No. 18 of the use of ICT in the curriculum (see Report No. 18, Chapter 2, Section: *Digital Tools*).

In terms of mathematics learning and development, when the intention is to consider the progress and activity of individual learners a social constructivist/cognitive perspective is helpful, but when the intention is to focus, on, for example, teaching practices, a sociocultural perspective is appropriate. Cobb and Yackel (1996) support this pragmatic view and emphasise the use of the perspective which is most helpful for the purpose:

> *The sociocultural approach…focuses on the social and cultural bases of personal experience, whereas analyses developed from the emergent [cognitive] perspective account for the constitution of social and cultural processes by actively cognizing individuals.* (p. 188)

By focusing on cognitive and sociocultural perspectives, we provide ourselves with different lenses with which to view mathematics learning and the pedagogy that can support it. Speaking of how theory is used to investigate and explain the complexity of human learning of mathematics, Lerman (1998) describes different perspectives as the zoom of a lens. The focus can be on mathematical tasks, representations and inscriptions, on problems and problem-solving, on the individual or the group, on the interactions between them, on communication and gesture and all the contexts in which these occur.

Sociocultural Perspectives

Sociocultural theories emphasise the social and cultural as inseparable contexts in which learning can be understood. They are sometimes referred to as cultural-historical theories, in order to explain the role that the past is seen to play in present culture and in social interactions. Sociocultural theories are increasingly the dominant framework used in early childhood education to explain young children's learning (NCCA, 2009a). Sociocultural theories include the range of Vygotskian and post-Vygotskian theories. Vygotskian theory argues that learning is socially mediated from the beginning. Notions such as 'interactions', 'shared attention' and 'intersubjectivity' are crucial. Bodrova and Leong explain that 'A mental function exists or is distributed between two people before it is appropriated and internalised*'* (2007, p. 79). Shared activities and shared talk are essential contexts within which learning occurs. Key sociocultural theorists such as Rogoff and Bruner also take a sociocultural approach to learning.

Rogoff (1998, p. 691) describes learning or development as a transformation of participation. From her perspective, transformation occurs at a number of levels: for instance, the learner changes at the level of their involvement, in the role they play in the learning situation, in the ability they demonstrate in moving flexibly from one learning context to another, and in the amount of responsibility taken in the situation. Learning is seen as a process by which children change as a result of taking part in activity. They become more able, and they participate with increasing confidence in similar activities. Children change both in their understanding of the activity and in terms of their role in the activity. Rogoff emphasises the personal, interpersonal and community aspects of the learning situation. The community aspect draws attention to culture, the interpersonal aspect draws attention to the interactions that are part of the learning process and the personal aspect draws attention to transformations in individuals' participation in activity. This perspective is coherent with Bruner's views.

Bruner's (1996) sociocultural theory of learning suggest that the process of learning is as much a social construction as it is an individual one – 'human mental activity is neither solo nor conducted unassisted, even when it goes on 'inside the head' (p. xi). In his view, culture shapes minds as 'it provides us with the toolkit by which we construct not only our worlds, but our very conceptions of ourselves and our powers*'* (p. x). In seeking to understand learning, Bruner argues that

> *…you cannot understand mental activity unless you take into account the actual setting and its resources, the very things that give mind its shape and scope. Learning, remembering, talking and imagining: all of them are made possible by participating in a culture.* (pp. x-xi)

Agency, collaboration, reflection and culture are four crucial ideas for learning identified by Bruner. He emphasises the role of language in the functioning of the mind and school as a culture itself, not just a preparation for it. He sees interactions between the learner and more experienced others as crucial to learning. More experienced others scaffold learning. The tools, physical and cognitive, that are used by people to assist in making and sharing meaning are considered by Bruner (1966) to be highly significant in determining learning. Some tools enhance action, others enhance the senses while still others enhance thought. The expectation is that highly abstract uses of symbolic forms and language – both spoken and written – are generally developed in schools.

Learners appropriate or internalise cultural tools (e.g., language, computers, numbers) to their own activity. Internalisation means 'knowing how', while appropriation means taking a tool and making it one's own. However, in terms of learning mathematics, this doesn't mean appropriation of the ideas of others: rather it means learners gradually transform initial ideas into fully-developed mathematical concepts under the influence of interaction with adults. From the sociocultural perspective, there is a back and forth relationship between notations-in-use and mathematical sense making, *'*cultural conventions such as notational systems…shape the very activities from which they emerge, at the same time that their meanings are continuously transformed as learners produce and reproduce them in activity*'* (Meira, 1995, p. 270). In early childhood, children initially develop their own marks and representations to communicate their mathematical thinking. These mathematical graphics can include, for example, scribbles, drawings and invented symbols and perhaps numerals and letters. Critically, these lay down the foundations for the later use of standard forms of written mathematics (Carruthers & Worthington, 2006). Perry and Dockett (2008) suggest that children develop their own symbol systems first, and use this knowledge until another, more standardised system, can be taken on board. Interactions with more knowledgeable others are particularly important since it is as a result of interactions about the meanings of marks and symbols (their own and the more conventional ones) which enable children to learn about the meaning and roles of mathematical symbols. This can be compared with encouraging young children to use their own strategies and methods to solve mathematical problems. Hence, children can be encouraged to use their own language, at least in stages where their concepts are being formed.

A Cultural-Historical Activity Theory Perspective

Activity theory is a development of aspects of Vygotsky's work (e.g., Engerström et al., 1999). Modern developments of activity theory are known as cultural-historical activity theory (CHAT) and these are characterised as a framework rather than as a theory with a set of neat propositions (Roth & Lee, 2007). Activity theory has been influential, particularly in relation to language, language learning and literacy but its implications for mathematics learning are only now being articulated. The framework focuses on culture, diversity, multiple voices, communities and identity (Ryan & Williams, 2007). It focuses on the joint activity in the learning situation, rather than on individual learners: 'a communal activity shared by a group typically has a communal 'object'. In schooling we might say the object is the 'task' to be carried out by the children and teacher*'* *(*p. 162). Activity theorists claim that making activity the focus results in a holistic view of learning (e.g., Roth & Lee, 2007, p. 218). Children use tools such as language, a particular action or resource to mediate knowledge in interactions with others. Ryan and William see potential in the way CHAT helps us to view the relationships between everyday activity and school mathematics and the role that everyday mathematics can play as a boundary object between the two. It also has potential for offering opportunities for shared learning and for the analysis of how this affects individual learning (Roth & Lee, 2007).

A Situative Perspective

Situative cognition sees understanding as situated in the body, in space and time, as well as socially and culturally. For instance, Ryan and Williams (2007) describe how situative theorists (e.g., Lakoff & Nunez, 2000) have analysed the number line from an embodied cognition point of view and how those authors see the number line as a particularly powerful model in that 'it allows the learner to situate themselves bodily and spatially in the mathematics in a powerful way' (p. 19). For example, young children can explore number relations and operations on a floor number line, by moving themselves forward and back on the line. Studies of out-of-school learning have revealed the situated nature of mathematical practices and of mathematical learning. For example, Nunes, Carraher and Schliemann (1993) have compared Brazilian children's facility with 'street mathematics' with their achievement in 'school mathematics'. From the situative perspective, learning takes place in the same context in which it is applied. This implies that it is important to think about the context in which learning takes place, all the constraints and affordances governing the site of learning and the use the learner makes of these (Greeno, 1991; 1997). When situated cognitionists speak of context, they are referring to a social context, defined in terms of participation in social practices (Lave, 1988). The social engagements that enable learning are a key focus. A number of studies in mathematics learning have indicated that different forms of mathematical reasoning arise in the context of different practices (e.g., Cobb & Bowers, 1999). The implication of this is that, if educators wish to encourage children's argumentation and reasoning, attention must be paid to the practices that are put in place to support these processes (see Report No. 18, Chapter 3, Section: *Mathematical Processes*).

Two theorists have worked separately (Lave, 1988; Wenger, 1998) and together (Lave & Wenger, 1991) to conceptualise a theory of learning which has given rise to notions of learning by belonging. They introduce the notion of legitimate peripheral participation as a pathway to learning in a community of practice. The practices of the community constitute what is to be known, learning is about participating more fully in the practices and moving from the periphery to the centre of practice (becoming more able). The idea that 'developing an identity as a member of a community and becoming knowledgeably skilful are part of the same process, with the former motivating, shaping and giving meanings to the latter, which is subsumed' (Lave, 1988, p. 65) can be used as a way of thinking of classrooms as mathematics learning communities.

To summarise, there are a number of implications for early mathematics education arising from sociocultural theories. For instance, interaction and collaboration with others is central. Culture plays a key role in learning; both the culture the children bring to the setting and the culture of the setting. This provides the context for learning. Children's agency is recognised, as is their strong interest in dialogue and discourse with others. Collaborating and establishing joint understanding are important. Establishing a zone of proximal development, within which to guide and support learning is a key task for the proactive educator. As well as scaffolding learning, the educator engages in the co-construction of meaning with the child, based on awareness and understanding of the child's perspective. Preschools and classrooms are seen as communities of practice where children learn mathematics as they engage with their teachers and peers in everyday activity in these settings.

Cognitive Perspectives

Cognitive theorists focus on internal cognitive structures and view learning as changes in these structures. Cobb (2007) emphasises cognitive psychologists' interest in how change occurs, most significantly qualitative changes in learners' mathematical reasoning. He identifies two general types of theories within the cognitive science tradition that relate to specific domains: theories which offer insights into the processes of children's learning and theories of the development of children's reasoning.

Constructivist Perspectives

Most of the current theorising about mathematical learning and development is grounded in Piaget's constructivism, a theory which emphasised the active construction of knowledge by learners through processes of assimilation and accommodation, in interaction with the environment. During the 1970s and 1980s, the Piagetian influence on mathematics education was enormous (Anderson, Anderson, & Thauberger, 2008). Through these decades various forms of constructivism were developed and there were ensuing conflicts, challenges and efforts at synthesis. Fosnot (1996), drawing from the work of various theorists, defines constructivism as

> *a theory about knowledge and learning; it describes both 'knowing' and how one 'comes to know'. Based on work in psychology, philosophy and anthropology, the theory describes knowledge as temporary, developmental, non-objective, internally constructed, and socially and culturally mediated. Learning from this perspective is viewed as a self-regulatory process of struggling with the conflict between existing personal models of the world and discrepant new insights, constructing new representations and models of reality as a human meaning-making venture with culturally developed tools and symbols, and further negotiating such meaning through cooperative social activity, discourse and debate.* (p. ix)

The use of the metaphor of learning as a process of construction has been traced from Vico's 18th century philosophical writings, to those of Kant in the 19th century. More recently, theorists such as Von Glasersfeld (1984) and Steffe (1992) were seen as radical constructivists due to their more radical views of learning when compared with those of Piaget. From the perspective of radical constructivists, learning is seen as self-regulation and self-organisation (e.g., Hufferd-Ackles, Fuson & Sherin, 2004). Since radical constructivism rejects the notion of an external, independent, objective reality, one aspect of individual learners' organisation is the world they construct through their experience, i.e. individuals construct their own ways of knowing (Von Glasersfeld, 1989).

Another important form of constructivism is the social constructivism of Ernest (1991) and colleagues. This is based on three grounds: linguistic knowledge, conventions, and rules form the basis for mathematical knowledge; interpersonal social processes are needed to turn an individual's subjective knowledge into accepted objective knowledge; objective knowledge is understood to be social (Sriraman & Haverhals, 2010). Socioconstructivists see as complementary the social and cognitive aspects of knowledge construction, explaining learning by drawing from both perspectives. Differences in the various forms of constructivism essentially revolve around the interplay between subjective and objective knowledge (Sriraman & Haverhals, 2010).

The psychological constructivist view of how children learn mathematics is, according to Battista (2004):

> *determined by the elements and organisation of the relevant mental structures that the students are currently using to process their mathematical worlds…To construct new knowledge and make sense of novel situations, students build on and revise their current mental structures through the processes of action, reflection and abstraction.* (p. 186)

This conception of learning mathematics is the one which underpins the learning trajectories literature which is reviewed in *Chapter 5* in this report.

Various attempts have been made to derive teaching approaches coherent with constructivist perspectives. For instance, Jaworski (1992) proposed three elements inherent in constructivist mathematics teaching: the provision of a supportive learning environment; offering appropriate mathematical challenge; and nurturing processes and strategies that foster learning. Constructivist teaching techniques are sometimes associated with 'discovery methods' and often contrasted with the explicit presentation of information to learners (e.g., Sweller, 2009). One critique of constructivist approaches is that they offer minimal guidance to learners (e.g., Kirschner, Sweller, & Clark, 2006), but this is disputed by proponents of such approaches. Duffy (2009) argues that in fact the difference in constructivist and explicit instruction approaches resides not in how or indeed how much guidance they offer to learners, but in their conception of the stimulus for learning. He considers that this is not addressed in explicit instruction approaches but in contrast is seen as central in constructivist approaches. That stimulus for constructivists is the need for learners to understand, to make sense of what it is they encounter.

Constructionism is a theory of learning which takes cognisance of the role of cultural tools, while also building on constructivism and sociocultural theories. Below we explore how this perspective underpins recent developments in digital learning and in the use of digital tools for learning.

Constructionism

The core concern of sociocultural theories is the mediated nature of all human activity through interactions with others around tasks and activities and with material and symbolic tools. From this perspective, 'tools' are conceived in a broad sense, including not only physical artefacts but also symbolic resources such as those of natural language and technical procedures such as mathematical algorithms. Cultural tools, whether physical or symbolic, are considered to influence the ways in which people interact with and think about the world. Bruner (1973, p. 22) saw thinking as the 'internalisation of 'tools' provided by a given culture' while Vygotsky (1978) saw changes in tools as bringing about changes in thinking, with these changes in turn associated with changes in culture.

Digital technologies are the cultural tools of today's digitised society. Their role as mediators of human learning is increasingly more complex when one considers the range and scope of computational tools currently available. As mediating tools, they function as intellectual partners with learners in order to enable them to think in ways that otherwise they would not or could not. They amplify, extend and enhance human thinking processes, thus offering a cognitive tool to engage and facilitate cognitive and metacognitive processing (Jonassen, Peck & Wilson, 1999). Jonassen (1996) uses the term 'mindtools' to highlight the power of digital technologies to support knowledge construction and critical thinking. Building on the concept of distributed cognition (Salomon, 1993), he argues that digital technologies should not support learning by attempting to instruct learners but rather should be used as knowledge construction tools that students can learn *with,* not *from*. In this way, learners can be perceived as designers, using the technologies as tools for analysing the world, accessing information, interpreting, organising and constructing their personal knowledge, and representing what they know to others (Jonassen & Reeves, 1996; Jonassen, Peck & Wilson, 1999).

Constructionism is a theory of learning which takes cognisance of the role of cultural tools, while also building on constructivism and sociocultural theories. Papert (1993), who worked with Piaget in the late 1950s and early 1960s, developed this theory of learning based upon Piaget's constructivism. He states:

> *constructionism, my personal reconstruction of constructivism, has as its main feature the fact that it looks more closely than other -isms at the idea of mental construction. It attaches special importance to the role of construction in the world as a support for those in the head, thereby becoming less of a purely mentalist doctrine.* (p. 143)

Papert and Harel (1991, p. 1) further explain how constructionism relates to constructivism with the statement that 'the N word as opposed to the V word – shares constructivism's connotation of learning as 'building knowledge structures'. Learners, consequently, are understood as active builders of their own knowledge and learn with particular effectiveness when they are engaged in constructing personally meaningful artefacts. However, constructionists argue that learning 'happens especially felicitously in a context where the learner is consciously engaged in the construction of a public entity whether it's a sand castle on the beach or a theory of the universe' (ibid, 1991, p. 1).

In this sense, constructionism 'is interested in how learners engage in a conversation with [their own or other people's] artefacts, and how these conversations boost self-directed learning, and ultimately facilitate the construction of new knowledge' (Ackermann, 2001, p. 85).

In our digitised society, from a mathematical perspective, artefacts can include designing and building computer programs, databases, animations or robots. These artefacts are 'objects to think with' (Papert, 1980, p. 12; Turkle, 1995). Through their use, learners are enabled to manipulate and reflect on what they know, and use these reflections to further construct knowledge (Reeves, 1998). They are also a means by which others can become involved in the thinking process. The learner's thinking benefits from interaction with others as the multiple views and discussions that result from such interactions are the greatest source of alternative views needed to stimulate new learning (Von Glasersfeld, 1989). In this way, learners become more engaged in constructing personal and socially-shared understandings of the phenomena they are exploring (Jonassen & Carr, 2000). It follows that the tools and materials used influence the nature of the artefact and therefore the thinking. According to Butler (2007, p. 64), 'There is consequently an interrelatedness of a symbiotic nature that exists between learners, the materials they use and the constructed artefact that they create'. This becomes their 'object to think with'.

Using digital technologies to construct personally meaningful artefacts enables learners to design their own representations of knowledge rather than absorbing representations preconceived by others. As stated by Jonassen and Carr (2000), they 'engage learners in a variety of critical, creative and complex thinking such as evaluating, analyzing, connecting, elaborating, synthesising, imagining, designing, problem-solving and decision making' (p. 168). As such, children not only engage more deeply with content but they can also access powerful mathematical ideas hitherto considered not possible. For example, dynamic geometric software (DGS) programs are tools that can be used to construct and manipulate geometric objects and relations (Battista, 1998; Healy & Hoyles, 2001). Erbas and Aydogan Yenmez (2011) claim that DGS has great potential to impact the teaching and learning of school geometry, particularly if used in a reflection-centred and problem-solving based learning environment. According to Battista (2001), DGS enables children to 'develop rich mental models' which help them 'to reason in increasingly sophisticated ways' (Battista, 2001, p. 118) moving them 'to higher levels of geometric thinking' (Olkun, Sinoplu, & Deryakulu, 2005, p. 11).

To illustrate this, a triangle constructed using DGS will not be a static triangle fixed in space. It can be manipulated to make any desired triangle that fits on the screen, no matter what its shape, size or orientation (Forsythe, 2007). By constructing different triangles and observing the changes in a dynamic manner, the learner is exploring the properties of shape and is not confined to the use of textbooks and commercial sets of 2-D shapes which tend to reinforce visual prototypes (Pengelly, 1999; Frobisher, Frobisher, Orton, & Orton, 2007). A reliance on visual prototypes is characteristic of those operating at Level 0 on the van Hiele geometric reasoning levels. Seventy percent of students leave primary education with a dominant geometric reasoning level of '0' (Battista, 1998) instead of a recommended level two (Van de Walle, Karp, & Bay-Williams, 2010). However, while these tools have the potential to transform 'mental functioning in fundamental ways' (Chu & Ju, 2010, p. 65), it is imperative that they are used in learning environments that encourage thoughtful reflection (Hannafin et al., 2001; Reynolds & Harel-Caperton, 2011). Consequently, a key role of teachers is to foster the development of a reflective culture in their classrooms (McKenzie, 1998). Thus, constructionism provides a particular perspective on how the use of digital tools impacts children's mathematical thinking and reasoning and promotes the development of their understandings.

A Redeveloped Primary School Mathematics Curriculum

The 1971 mathematics curriculum, *Curaclam na Bunscoile* (Department of Education, 1971), drew heavily on Piagetian ideas, in particular on stage theory. The more recent PSMC (Government of Ireland, 1999) espoused a social constructivist view as evidenced in the emphasis on the social aspects of learning. As discussed in the Introduction to this report, when it was introduced in 1999, the PSMC was well received. While maintaining some important links with the 1971 curriculum, it also drew heavily on Vygotskian ideas about teaching and learning, in that it emphasised the social aspects of mathematics development, the importance of language in acquiring mathematical knowledge, and the key role of the teacher in modelling and supporting children's emerging understanding of mathematics. However, the role of the mathematics curriculum in the minds of teachers is an issue that needs some thought. The issue is how the theoretical underpinnings of the curriculum are commensurate with classroom practice. There is ample evidence that textbooks are used as the main planning tool for the teaching of mathematic in many classrooms (e.g., Eivers et al., 2010; Dunphy, 2009). The design of textbooks, which include pages of repetitive work with barely discernible levels of ascending difficulty (e.g., the repeated practice of addition of two digit numbers without 'carrying' followed later by addition 'with carrying' is at variance with the emphases suggested in the current chapter). Similarly, an understanding of mathematics as largely symbolic and the learning of mathematics as the manipulation of symbols is not coherent with, for instance, the embodied stance of Lakoff and Nunez (2000). An embodied stance is where an idea is expressed or represented physically or concretely. It assumes that young children often communicate and articulate their understandings and ideas by using actions and gestures instead of/as well as words. It might be claimed that the predominance of coloured pictures in current mathematics textbooks has been influenced by 'situated learning' theories, where context is an important basis for learning mathematics. However some of these have been critiqued by Charalambous, Delaney, Hsu and Mesa (2010). Their findings in relation to the addition and subtraction of fractions are that:

> *The Irish textbooks differed from those in the other two countries [Cyprus and Taiwan] in terms of the context around which the worked examples were built. Most worked examples in the Irish texts were set in exclusively mathematical contexts…In the other two countries, worked examples were more often embedded in 'real-world' contexts…Irish textbooks had the greatest number of 'completed' worked examples…all Irish worked examples explicitly illustrated the steps to be followed when completing procedures.* (p. 135)

The above authors argued that there is a need to examine textbooks in order to understand differences in teaching approaches and achievement in different countries. The relevance of this for the mathematics curriculum for children aged 3–8 is that a redeveloped curriculum needs to consider how the range of resources that support pedagogy cohere with the theoretical stance of the curriculum.

Implications for Practice

Table 2.1 below outlines the key implications of the perspectives for learning, teaching curriculum and assessment. Of necessity, these are generalisations. It is important to note that there can be differences in interpretations in relation to the various perspectives, in particular the sociocultural perspectives. This arises from the fact that sometimes theorists who see themselves as located in slightly different places theoretically often use similar concepts and language to articulate their positions (e.g., Ryan & Williams, 2007). This makes their perspectives at times difficult to distinguish.

Table 2.1. Key Implications of Theoretical Perspectives

	Cultural-historical activity theory	Situative
Emphases	▪ The structure of activities ▪ Activity as continually negotiated between participants with the resources of their environments ▪ Tools can be either material or conceptual	▪ The larger systems: includes people, interactions and all the elements of the environment ▪ Practices of the community ▪ Becoming more central in a community's practices
Learning	▪ Learning is the result of everyday practice and processes of meaning-making ▪ An expansive view of learning ▪ Zone of proximal development is a key concept	▪ Learning is a change in participation…about becoming more centrally involved in the practices of the community ▪ Changing forms of participation are part of a process that shape identity formation ▪ Diversity is the expectation: learning more multi-path in nature ▪ Interpretation of artefacts such as symbols and icons is a crucial part of social practices

Table 2.1. Key Implications of Theoretical Perspectives *(continued)*

	Cultural-historical activity theory	Situative
Teaching	▪ Use of tools (for example, technology or symbols) as mediators in activity	▪ The focus is on the group of learners ▪ Dialogical pedagogy of argumentation and discussion designed to support effective conceptual learning ▪ Identification of conceptual obstacles ▪ Scaffolding learning using models ▪ Focus on developing mathematical skills within the context of real-world learning situations ▪ Work with 'rich' mathematical problems e.g. problem-based learning ▪ Foster a community of learners ▪ Foster the development of learner identity ▪ Foster metacognitive awareness ▪ Teach at upper levels of ZPD
Curriculum	▪ Tools can be material or conceptual	▪ Focus on processes with an emergent view on content ▪ Mathematics situated in curriculum tasks which use cultural tools ▪ Mathematical activities must make sense and be part of a child's larger social activity ▪ Models and representations used to solve practical problems
Assessment	▪ Expectations of difference	▪ Assessment of participation in meaningful activities ▪ Diagnosis of errors since these indicate intelligent constructive activity

Table 2.1. Key Implications of Theoretical Perspectives *(continued)*

	Social Constructivist	Constructionism
Emphases	▪ The social context ▪ Interpersonal relations, especially teacher-learner and learner-learner interactions ▪ Negotiation, collaboration, and discussion ▪ The role of language	▪ Constructions in the world as supports for constructions in the head ▪ Tools, media and contexts ▪ Artefacts as objects to think with ▪ Learners construct knowledge particularly well when constructing personally meaningful entities ▪ Learners' reflections and social expression about their work in progress…in a community of practice ▪ Tool use has the potential to transform mental functioning in fundamental ways when combined with thoughtful reflection on the learning process
Learning	▪ Learning is a change in understanding/thinking ▪ Focus on qualitative changes in reasoning ▪ Importance of children reflecting on their work	▪ Learner sets their own learning goals ▪ Emphasises the idea of diversity, recognises that learners can make connections with knowledge in many different ways ▪ Encourages a variety of learning styles and representations of knowledge ▪ Intimate connection between knowledge and activity ▪ Active process that involves individuals asking questions, discussing and solving problems, sharing ideas, thinking critically and exploring and assessing what they know.
Teaching	▪ The focus is on individual learners ▪ Teacher modelling important ▪ Present cognitive challenge ▪ Strategic learning encouraged ▪ Encourage self-regulation of learning	▪ Learning environment fosters discussion and reflection ▪ Learning environment designed to provide opportunities for inquiry-based explorations, collaboration and reflection using a range of computational tools ▪ Foster self-regulation ▪ Foster the development of a reflective culture ▪ Foster culture of collaboration among peers ▪ Reflection/articulation ▪ Foster meta-cognitive awareness ▪ Teacher, or knowledgeable other, participating in the learning process alongside the learner, cueing, prompting, questioning where necessary

Table 2.1. Key Implications of Theoretical Perspectives *(continued)*

	Social Constructivist	Constructionism
Curriculum	▪ Focus on conceptual understanding ▪ Tasks/activities are incremental and build on what children already know ▪ Artefacts used serve to influence thinking	▪ Meaningful, authentic activities that help the learner to construct understandings and develop skills ▪ Long term problems/projects related to the learner's needs and interests ▪ Authentic relevant real-world problems ▪ Learning to learn/thinking about thinking
Assessment	▪ Problem-focused ▪ Authentic tasks focused on a wide range of cognitive behaviours (lower and higher order) ▪ Aimed at eliciting expertise	▪ Encourage learners to make predictions and to constantly reflect on discrepancies between their predicted answers and those found. As they do so they refine their theories and understandings.

In the United States, cognitive science emphases are reflected in many high-profile statements (e.g., NRC 2001, 2005; NCTM, 2000). They are also reflected in the work of prominent early childhood mathematics educators (e.g., Clements, Sarama & DiBiase, 2004). Meanwhile, in countries such as Australia there has been a movement amongst mathematics educators and in curriculum policy towards socioculturally-oriented approaches to teaching, learning, assessment and curriculum. See for instance Conway and Sloane's (2005) account of changes in assessment practices in Victoria and New South Wales. See also Perry and Dockett's (2008) articulation of a socioculturally oriented mathematics curriculum at preschool level, first presented as early as 2002.

The PSMC (Government of Ireland, 1999) can be seen as having a socio-constructivist orientation which had its roots in Piagetian/radical constructivism, though there are also some adherences to a Vygotskian perspective. Social constructivism has two formulations, one with its roots in Piagetian/radical constructivism, and the other with its roots in Vygotskian theory (Ernest 2010, p. 54). We consider this distinction helpful in considering how the theoretical orientation of a redeveloped curriculum for the mathematics education of children aged 3–8 years might be distinguished from that of the 1999 PSMC. A new iteration of the curriculum which takes account of the sociocultural perspectives described above would be much more firmly rooted in recent theories developed from a Vygotskian base and which emphasise children's participation in mathematics, their identity as mathematics learners, and their interactions in communities of learners.

Conclusion

In early childhood mathematics education sociocultural/cultural-historical theories are of particular importance, given their capacity and usefulness for explaining early learning and the role of cultural and social influences in learning. Recent versions of constructivism help to explain the mechanisms of learning and these are central to a comprehensive theory of early mathematics learning. The insights afforded by considering the cultural and social dimensions of the learning situation, including cultural tools and media, explain what children learn, why they learn in particular circumstances and how they learn. They also indicate clearly how early mathematical learning and development can best be supported. An explanatory framework recognising the role of internal processes, but foregrounding the fact that mathematics learning and development are dependent on children's active participation in social and cultural experiences, provides the basis for a powerful theoretical framework for mathematics education for children aged 3–8 years. Important too we feel are the insights offered by the Realistic Mathematics Education (RME) approach. However, we have left our discussion of that until Chapter 5 (Section: *Developing Children's Mathematical Thinking: Three Approaches*) since RME is an approach to mathematics education, rather than a 'grand theory' of learning.

The key messages arising from this chapter are that

- Cognitive and sociocultural perspectives provide different lenses with which to view mathematics learning and the pedagogy that can support it. Cognitive perspectives are helpful in focusing on individual learners, while sociocultural perspectives are appropriate when focusing on, for example, pedagogy.
- Sociocultural perspectives, cognitivist perspectives and a constructionism perspective each offer insights which can enrich our understanding of issues related to the revision of the curriculum. They do so by providing key pointers to each of the elements of learning, teaching, curriculum and assessment. Used together, they can help in envisaging a new iteration of the PSMC.
- Learning mathematics is an active process which involves meaning making, the development of understanding, the ability to participate in increasingly skilled ways in communities of learners, and engagement in mathematization and the development of a mathematical identity.
- The proactive role of the teacher must be seen to involve the creation of a zone of proximal development, the provision of scaffolding for learning, and the co-construction of meaning with the child based on awareness and understanding of the child's perspective. It also involves a dialogical pedagogy of argumentation and discussion.

CHAPTER 3

Language, Communication and Mathematics

Language plays a critical role in developing young children's mathematical thinking (e.g., Ellerton, Clarkson & Clements, 2000; Whitin & Whitin, 2003). Talking about mathematical thinking and engaging in reasoning, justifying and argumentation are central to mathematics education for all children aged 3–8 years (Ginsburg, 2009a). According to the NRC report:

> *Children must learn to describe their thinking (reasoning) and the patterns they see, and they must learn to use the language of mathematical objects, situations and notation. Children's informal mathematical experiences, problem solving, explorations, and language provide bases for understanding and using this formal mathematical language and notation.* (2009 p. 43)

In his seminal work on mathematics register, Haliday (1978) argues that acquiring mathematics involves learning not just the vocabulary of mathematics, but also the styles of meaning, modes of argument, and methods of thinking mathematically. Similarly, Schleppegrell (2010) calls on educators to view mathematics as discourse. In this view planned activities provide opportunities to engage learners in such discourse, without losing a focus on the underlying mathematics. This perspective is consistent with sociocultural theories of mathematics learning which see children being enculturated into mathematics through social activity and discourse (see Chapter 2, Section: *Sociocultural Perspectives*). Perry and Dockett (2002) emphasise the value in allowing young children to use their own symbols and their own names for mathematical entities in the early stages of learning mathematics, followed by a gradual shift to more formal systems. They also draw attention to the challenges facing young children in settings where the discourse of mathematics involves a language that is different to the language of the home.

The term 'math talk' is often used to describe the language interactions that occur when children are supported in talking about their mathematical thinking, including their formal and informal representations of mathematical ideas and symbols. Indeed, the NRC report (2009) notes that a 'math-talk learning community', in which all children have opportunities to describe their thinking, has the potential to improve children's mathematical language and their general language levels. It also points to the importance of children using language to make connections across different domains of mathematics, and across mathematics, other learning areas, and everyday life.

The importance of oral language in developing mathematical understanding is recognised in policy statements and curriculum documents. For example, the NAEYC/NCTM (2002/2010) *Position Statement on Early Childhood Mathematics* (3–6 years) includes as a recommendation the active introduction of 'mathematical concepts, methods and language through a range of appropriate experiences and strategies' (p. 9), while taking children's cultural background and language into consideration. The same report notes the many opportunities that can arise to integrate mathematics with other learning activities (e.g., storytelling) which can support children in learning mathematical vocabulary. *Aistear* (NCCA, 2009a) includes, as one of the key aims of its *Communicating* theme, a broadening of children's understanding of the world by making sense of experiences through language, including mathematical language.

This chapter examines the role of language in learning mathematics. First, it looks at the relevance of language for learning different aspects of mathematics and the research that supports the use of mathematical language in children's homes, in the preschool and in primary school. Second, it looks at theories of communication in mathematics learning and links them to broader theoretical frameworks for learning mathematics that were considered in *Chapter 2*. Third, it describes the development of children's mathematical vocabulary in the context of broader conceptual development. Fourth, it considers groups who may struggle with language in general, and therefore may experience additional challenges in bridging the gap between informal and more formal mathematical ideas.

The Role of Language in Developing Mathematical Knowledge

There is a complex relationship between language development and growth in mathematical thinking. Even before they acquire language it seems that infants in their first year may be aware of changes in the numbers of items in small sets (e.g., Feigenson & Carey, 2005) and can discriminate between larger sets of items where the proportional difference is large (e.g., 8 vs. 16 items) (Brannon, Abbott & Lutz, 2004). In these early stages, there is a complex relationship between representation of number, and representation of associated variables such as area, size and arrangement of items. Moreover, such early number representations may work independently of the language system (e.g., Gelman & Butterworth, 2005). According to Zur and Gelman (2004), 3-year-olds can use basic number concepts to predict and check the results of additions and subtractions to sets of up to five items, even if they are unable to produce such sets by counting.

There is some disagreement among researchers as to when children integrate their number word knowledge (e.g., counting) with their non-verbal number systems. Carey (2004) suggests that language factors (including knowledge of plurals) can 'bootstrap' number development as they combine with earlier non-verbal representations of number, to provide a new and comprehensive number system. For example, children's knowledge of number word sequence, which may have been acquired initially without numerical meaning, combines with their representations of small sets of items. This combination is seen as providing a basis for symbolic representation of number.

Others (e.g., Rips, Asmuth & Bloomfield, 2008) argue that knowledge of the number word sequence is not sufficient to support conceptual development, and that it is only at a much later stage – called 'advanced counting' – that children can construct the next number term from any number in the sequence, based on the correspondence between the structure of the number sequence and the properties of natural numbers. This is evident in a study by Sarnecka and Carey (2008) in which children aged 2 years and 10 months to 4 years and 3 months were the subjects. While almost all children could produce the number sequence to 10, conceptual understanding varied considerably, with 40% of children showing no understanding that going forward in the number sequence corresponds to adding, and going back to subtracting.

Taken together, such studies suggest a need for early childhood educators to support young children in establishing a conceptual link between language (in this case, the number sequence) and understanding of number. According to Donlan (2009), the process of integrating procedures and concepts (e.g., rote counting and underlying principles of counting) is important.

Adult Support

As young children grow and develop so too does their familiarity with and use of language. Everyday situations both support and encourage children's use of mathematically-related language, especially where these involve interactions with adults.

There is evidence that the mathematical language used by adults in preschool settings can have an impact on children's mathematical knowledge. In a study involving 26 preschool teachers and their children, Klibanoff et al. (2006) recorded instructional time, including circle time, over a seven-month period for one hour per month in each class. Although few teachers led planned mathematics lessons during the recorded observations, many incorporated mathematical inputs in their speech. Children were pre- and post-tested on mathematical knowledge. Children in settings in which teachers used many instances of math talk were more likely to improve over the course of the study than children in settings in which less mathematical language was used. An interesting outcome of the study was the wide range of mathematical inputs across settings, ranging from 1 to 104 instances of mathematical utterances, with an average of 28. Forty-eight percent of all inputs were references to cardinality, while inputs relating to equivalence, non-equivalence, ordering, calculation and placeholding were much less common. This outcome of this study is consistent with the work of Dickinson and Tabors (2001), whose research with preschoolers showed that, during large-group activities, more frequent use of teachers' explanatory talk and use of cognitively challenging vocabulary were associated with better learning outcomes for children.

Familiarity with spatial language is particularly important in learning and retaining spatial concepts. Gentner (2003) found that children who heard specific spatial labels during a laboratory experiment that involved hiding objects ('I'm putting this on/in/under the box') were better able to find the objects than children who heard a general reference to location ('I'm putting this here'). Moreover, this was true even two days later, without further exposure to the spatial language (Loewenstein & Gentner, 2005). Szechter and Liben (2004) observed parents and children in the lab as they read a children's book with spatial-graphic content. They found an association between the frequency with which parents drew children's attention to spatial-graphic content during book reading (e.g., 'The Rooster is really tiny now') and children's performance on spatial-graphic comprehension tasks.

Levine et al. (2012) examined how parents use spatial language during puzzle play in a study in which parent-child pairs were observed for an hour during naturalistic interactions every 4 months from 26–46 months. Children who were observed playing with puzzles performed better on a mental rotation task at 54 months, after controlling for parent education, income and overall parent word types. Further, among those who engaged in playing puzzles during observations, those who played more puzzles did better. Although the frequency of puzzle play did not differ for boys and girls, the quality of puzzle play (a composite of puzzle difficulty, parent engagement, and parent spatial language) was higher for boys than for girls. In interpreting this, Levine et al. (2012) raised the possibility that girls might benefit from more complex puzzles. There is also evidence that higher amounts of parent spatial language occur during guided block play in which there is a goal than during free play with blocks (Shallcross et al., 2008). Thus, it is possible that spatial activities, spatial language, or both promote the development of spatial skills, such as block building and mental rotation.

Language is one domain-general cognitive skill on which young children may vary. Others include memory, visual-spatial skills, and executive functions (Mazzocco, 2009), though none of these are independent of one another and, like language, they are associated with learning difficulties in mathematics.

The Nature and Scope of Mathematical Discourse

Language plays as important a role in mathematics learning as in other school subjects (Schleppegrell, 2010). While teaching the vocabulary of mathematics to young children is important (e.g., Neuman, Newman & Dwyer, 2011), research has gone beyond the word level in identifying and describing the language challenges of mathematics. Haliday (1978), for example, refers to a mathematics register – 'the meanings that belong to the language of mathematics' (p. 79). In this sense, learning the language of mathematics does not entail just learning new words, but also learning new 'styles of meaning and modes of argument…and of combining existing elements into new combinations' (pp. 195–196). Hence, while activities such as counting and measuring may well draw on everyday language, children learning mathematics need to use language in new ways to serve new functions. According to Schleppegrell (2010), the concept of a mathematical register draws attention to the ways in which mathematical knowledge is different from knowledge of other academic subjects. She argues that learners need to be able to use language to participate effectively in 'ways of knowing that are particular to mathematics' (p. 79). Hence, if we view mathematics as discourse, we need to identify ways of apprenticing children into particular ways of doing mathematics in particular discursive contexts. Pimm (1991) argues that children in school are attempting to acquire communicative competence in the mathematical register, and that classroom activities should be carefully examined from this perspective in order to see what opportunities they offer for children's language learning. Silver and Smith (1996) point out that, in developing and using language in mathematics, it is important that mathematics does not get lost and that discourse focuses on 'worthwhile tasks that engage students in thinking and reasoning about important mathematical ideas' (p. 24).

A number of theories reviewed in Chapter 2 under the broad umbrellas of constructivist/cognitive and sociocultural can be drawn on to explain the relationship between language and mathematics:

- Cognitive theories, which have their origins in the work of Piaget, focus on the individual child's construction of internal representations or structures. According to Cobb and Yackel (1996), constructivist perspectives can be characterised as interpretive, since knowledge is actively constructed by children in interaction with their environment. Constructivists focus on the way children talk about mathematics to investigate their development of mathematical knowledge.

- Sociocultural theories focus on discursive practices and the interaction of children. They draw on Vygotskian frameworks that stress the interaction between language and cognition and highlight the social dimension of language and the role of communication and participation. In sociocultural terms, children are enculturated into mathematics through social and discursive activity.

Other researchers (e.g., Cobb, Yackel & McClain, 2000; Gutiérrez, Sengupta-Irving & Dieckmann, 2010) have built on cognitive and sociocultural theories to view language as a tool for thinking, interpreting, constructing knowledge and developing mathematical ideas. In this view, oral language is one of a range of resources that also include written language, gesture, symbols, equations, graphs and other visual representations. Hence, all of these need to be taken into account in interpreting how children construct meaning during mathematical activities. Children coming from different backgrounds and contexts may be positioned in different ways to use these resources. According to Schleppegrell (2010), differences should be acknowledged and viewed as resources in the mathematics classroom if the focus is on meaning, and if teachers are able to draw on different perspectives.

Sfard (2007) makes a useful distinction between language and discourse when she identifies language as a tool and discourse as an activity in which the tool (one of several) is used or mediates. For Sfard, knowing mathematics is synonymous with the ability to participate in mathematical discourse. Hence, learning is a special type of social interaction aimed at modification of other social interactions. An implication of this is that teachers can help modify children's everyday discourse into a more mathematical discourse. Interestingly, Gutiérrez et al. (2010) point out that Sfard's communicational approach to mathematics does not imply that children must first encounter a mathematical idea, use it, and then formalise it later into mathematical conventions (the 'learning with understanding' approach). Instead, Sfard proposes that an existing discourse of mathematics (e.g., thinking about big numbers or infinity) can be used to initiate children into a discourse of new objects.

Sfard's work can also help to clarify the distinction between everyday (colloquial or primary) discourse and literate (scientific or secondary) discourse. Sfard (2001) argues that everyday discourse does not naturally evolve into scientific (e.g., mathematical) discourse. This is because mathematical discourses are mediated by symbolic artefacts designed to communicate specific conceptual understandings of quantities (that is, symbolic mediation is a key characteristic of mathematical discourse). Since such discourse is often not a part of children's everyday discourse, secondary discourse requires explicit teaching (Sfard & Cole, 2003, cited in Gutiérrez et al., 2010).

Gutiérrez et al. (2010) point out the importance of viewing everyday discourse and scientific (here, mathematical) discourse relationally. This implies that, rather than everyday discourse being viewed as a pre-requisite for mathematical discourse, mathematical discourse can be viewed as arising from (and feeding back into) everyday discourse. The two discourse types can be viewed as operating side-by-side, each being invoked in different circumstances depending on the context involved. An implication of this perspective is that children's general language skills can develop as a result of participating in mathematical discourse.

O'Halloran (2005) has focused on the characteristics of successful mathematics discourse as it relates to other available tools: (i) the meaning potential of language, symbolism and visual images are accessed; (ii) the discourse, grammatical and display systems of each resource function integratively; and (iii) meaning expansions occur when the discourse shifts from one resource to another (p. 204).

Establishing a Math-Talk Culture

NicMhuirí (2011) points to some of the differences between the discourse of traditional mathematics lessons, and the discourse of mathematics lessons that seem to engage children in mathematical discourse. While the former are often characterised by 'repeated iterations of lower-level questions' (p. 320) or the IRF (invitation-response-feedback/evaluation) pattern, and dominated by teacher talk, the latter can include 'patterns of dialogue that involve making conjectures, and examining and justifying one's own mathematical thinking and the mathematical thinking of others' (p. 320). Although NicMhuirí's analysis of mathematics lessons focused on third to sixth classes, her outcomes may have implications for mathematics teaching more generally. In particular, she identifies less helpful patterns of discourse where

- teacher intervention focuses on the solution provided by children rather than their mathematical thinking
- there are lengthy teacher explanations between questions/dialogue
- learners are prompted to arrive at a correct answer, with the teacher sometimes taking on the cognitively-demanding aspects of the task, and, on other occasions, focusing children's attention on critical aspects of the problem, even if the children were expected to solve the problem on their own.

Although the pattern of interactions in the lessons analysed by NicMhuirí may have been justified on the grounds that they keep the lesson moving along towards an end-goal, important opportunities for engaging in mathematical dialogue, including mathematical reasoning, may be overlooked. This and similar work (e.g., Dooley, 2011) point to a need to support teachers to reflect on their classroom dialogue, and provide children with more opportunities to engage in mathematical thinking, along the lines described earlier. Indeed, the relative difficulty that children in Ireland, including those in second class, encounter with solving mathematics problems (see *Introduction*)

point to the urgency of promoting more interactive mathematical discourse in learning settings. Others (e.g., Hufferd-Ackles et al., 2004) provide a framework for establishing and developing math-talk learning communities in learning contexts.

It also seems relevant, in the context of supporting mathematical discourse in early learning settings, to draw attention to more general strategies for language development (e.g., Dooley, 2011; Shiel et al., 2012) that teachers can implement including

- following the child's lead
- mapping language to the child's focus of attention
- cueing/prompting and inviting further comment
- extending the topic by providing further comment
- use of repetition, recasts and expansions
- modelling correct use of vocabulary in sentences
- use of topic elaboration.

NicMhuirí's work also highlights the importance of teachers engaging children in discussing and solving problems among themselves. This is consistent with sociocultural theories of learning that emphasise the role of language in acquiring knowledge in social communities, and with more general theories of learning mathematics that emphasise the role of argumentation (e.g., Perry & Dockett, 2008).

Learning Mathematical Vocabulary

In earlier sections of this chapter, we noted the importance of vocabulary in establishing bridges between young children's early sense of number and spatial sense, and their later mathematics learning. While mathematical vocabulary can be taught in formal or semi-formal settings such as maths classes, it can also be taught informally. As noted above, there is research evidence linking the frequency of adults' use of mathematical vocabulary in informal activities such as playing with bricks or solving a puzzle/jigsaw that can impact on children's mathematical learning.

Efforts have been made to specify the mathematical vocabulary that young children should learn. For example, in the current PSMC, specific mathematical vocabulary which should be addressed is highlighted in the content objectives. In matching equivalent and non-equivalent sets, children should be supported in learning terms such as *more than, less than, enough* and *as many as.* In developing spatial awareness, such terms as *above*, *below, near, far, right* and *left* are identified as a focus of instruction. In the United Kingdom, in support of the *National Mathematics Strategy*, the UK Department for Education and Employment (DfEE, 2000) issued a booklet for teachers that specified the range of vocabulary to be taught at each class level from reception (age 5 years) to year 6. In reception year, the mathematical areas under which vocabulary items are grouped include: counting and recognising numbers; adding and subtracting; solving problems; measures; shape and space; instructions and general. Importantly, the vocabulary booklet notes that key terms should, where possible, be taught in context, and instruction should be supported by the use of relevant real objects, mathematical apparatus, pictures and diagrams. The use by teachers of questions (both open-ended and closed) that enable children to use new vocabulary is stressed, and teachers are urged to be sensitive to the possibility that some vocabulary terms may be well understood by children in non-mathematical contexts or everyday language, but not in contexts where more precise mathematical understanding is important. In addition to targeted teaching of key vocabulary, opportunities should also be sought to support children's learning and the use of mathematical vocabulary in a range of contexts including play, mathematics lessons (e.g., when solving problems), and other learning areas.

Some researchers working with socio-economically at-risk preschool or kindergarten children (e.g., Neuman, Newman & Dwyer, 2011) have drawn attention to how such children often lack the conceptual knowledge required to understand mathematical discourse, and may need a more intensive approach to vocabulary development, compared with children who are not at risk. They report on a year-long programme administered to 3- to 4-year-olds in US Head Start classrooms that focused on word knowledge and conceptual development through taxonomic categorisation (categorising words) and embedded multimedia. Children in the programme, which covered aspects of health education (50 words) and living things (80 words) as well as mathematics (geometric shapes and number) (50 words), outperformed their counterparts in a control group on a range of outcome measures including domain-specific knowledge. Moreover, gains in word and categorical knowledge were sustained six months later. The authors interpreted the findings as suggesting that teaching words within taxonomic categories 'may act as a bootstrap for self-learning and inference generation'. The programme made a distinction between the concepts to be taught (e.g., some geometric shapes have corners, and some do not) and the target vocabulary words (e.g., specific shapes), with an instructional emphasis on both.

Variation in Language Skills and Impact on Mathematics

A number of groups are known to struggle with general language acquisition, including children living in disadvantaged circumstances, children who speak a language other than the language of instruction at home, and children who have a language impairment (see also the discussion in Chapter 6, Section: *Immersion Settings*).

We know that children living with disadvantage do not lack fundamental mathematical ability and that these children demonstrate few if any differences in the everyday mathematics they use in free play (e.g., Ginsburg et al., 2008). Familiarity with mathematics language is generally recognised as a key issue that must be addressed in early childhood mathematics education (e.g., Ginsburg, 2009a;

Hughes 1986). Mathematical language includes vocabulary, but just as crucial are language skills that enable the communication of mathematical thinking. The urgency to ensure that children living with disadvantage have adequate language experiences around mathematics is also emphasised by Perry and Dockett (2008) who argue that, 'without sufficient language to communicate the ideas being developed, children will have the opportunities for mathematical development seriously curtailed' (p. 93). Furthermore, they contend that the development of mathematical language, especially among non-English speakers, is particularly problematic because of mathematics' specialised vocabulary and because common words have specialised meanings.

In the United States, approximately 7% of children have specific language impairment (SLI), and while there is considerable variation within this group, many experience difficulty in learning the number-word sequence (Donlan, 2009). In one study, 5-year-olds with SLI were able to recite the number sequence up to 6, while their non-SLI counterparts reached 20 (Fazio, 1994). However, contrary to expectations, the SLI group showed a good understanding of the logical principles in object counting, including the principle that the final count word indicates the value of the set. When Fazio retested the children with SLI at 2-year intervals, they struggled on measures of basic calculation (Fazio, 1996, 1999).

A similar pattern of procedural weakness and conceptual strength emerged in a study of 7-year-olds with SLI. Forty percent of the group were unable to count to 20, whereas just 4% of typically developing 5- to 6-year-olds were unable to do so. Again, the performance of the children with SLI on a test of understanding of arithmetic principles was similar to typically-developing peers (Donlan, Cowan, Newton & Lloyd, 2007).

Nevertheless, Donlan (2009) warns that it is incorrect to accept that the effects of language difficulties on mathematical development are delimited in a clear way, with non-verbal number processes relatively unaffected. He points to a need for additional research that highlights how factors underlying SLI might impact on SLI children's performance on tasks of enumeration and calculation.

Conclusion

Language plays a key role in the development of children's mathematical thinking. Cognitive/constructivist and sociocultural theories of learning (see *Chapter 2*) support a strong focus on the use of language to acquire mathematical knowledge, and adults – whether parents, carers or teachers – are seen as key agents in supporting children's development of mathematical language across a range of informal and more formal contexts. While some of the mathematical language used in preschool and early school settings will be informal and will arise from children's participation in everyday activities (e.g., counting the number of children in a group, matching coats to children), other instances of language use will be planned around specific activities such as block building, solving puzzles/jigsaws, shopping or using mathematical software. These provide significant opportunities to introduce relevant mathematical vocabulary, engage children in using mathematical language through asking open or closed questions, paraphrasing or extending children's responses, and encouraging them to explain their thinking. Most importantly, children should be provided with opportunities to engage in mathematical talk with other children.

The key messages arising from this chapter are as follows:

- Cognitive/constructivist and sociocultural perspectives on learning emphasise the key role of language and dialogue in supporting young children's mathematical development. Emerging learning theories point to the importance of mathematical discourse as a tool to learn mathematics.
- In addition to introducing young children to mathematical vocabulary, it is important to engage them in 'math talk': conversations about their mathematical thinking and reasoning.
- Research indicates an association between the quality and frequency of mathematical language used by carers, parents and teachers as they interact with young children, and children's development in important aspects of mathematics. This highlights the importance of adults modelling mathematical language and encouraging young children to use such language as they engage in dialogical reasoning. Children's conversations among themselves about mathematical ideas can also support their development of mathematical knowledge.
- Children at risk of mathematical difficulties may need additional, intensive support to develop language and engage in mathematical discourse. In this context, extensive care and attention should be given to the language element of the learning and teaching of mathematics and extra supports should be provided in these contexts.

CHAPTER 4

Defining Goals

In Chapter 2 we saw how theories of mathematics learning have moved away from seeing learning as acquisition of knowledge towards seeing learning as the understanding of the practice of doing mathematics. This change in perspective implies the need for new learning goals for mathematics education. These new goals need to emphasise understanding and thinking as well as skills and facts. The specification of goals is an issue that is closely linked to pedagogy since different practices support different goals (Gresalfi & Lester, 2009). Awareness of the need to balance process and content goals is evident in a recent characterisation of early mathematics education in the United States (e.g., NRC, 2009). This focus is encapsulated in the following statement from Clements, Sarama and DiBiase (2004):

> *As important as mathematical content are general mathematical processes such as problem solving, reasoning and proof, communication, connections, and representation; specific mathematical processes such as organising information, patterning, and composing; and habits of mind such as curiosity, imagination, inventiveness, persistence, willingness to experiment and sensitivity to patterns.* (p. 3)

In this chapter we examine approaches to the specification of goals for early childhood mathematics education more closely. In identifying goals for young children's mathematics learning, commentators take different approaches and may choose to foreground particular goals. This is largely dependent on theoretical orientation, conceptions of mathematics, context and the age-range that they focus on. In this chapter we discuss one overarching framework related to higher-order thinking. We discuss two different approaches to the specification of goals for early childhood mathematics, one from a sociocultural perspective and one from a cognitive perspective. We consider how they deal with the content/process issue and we compare the approaches with that used for the specification of goals in the 1999 PSMC. We consider the implication for the structure of curriculum materials.

A Coherent Curriculum

The curriculum should have continuity from early childhood through all phases of education. One way of doing this and of mitigating discontinuities in mathematics learning is by having agreed goals, the nature of which can become more subject specific as children grow older (e.g., Pound, 1999). In Ireland, revised goals for mathematics education will need to build on the broad learning goals related to each of the themes of *Aistear* (*Well-being, Identity and Belonging, Communicating* and *Exploring and Thinking*). These themes provide the foundations on which subsequent mathematics-specific learning is based. In addition, the revised goals also need to be consistent with the goals of *Project Maths*.

Essentially, underlying views of mathematics, of knowledge and of learning are what determine the nature of the goals that are specified in the curriculum. In curricula, all of the elements, including theoretical orientation and goals, must align. In the final analysis, considerations related to early learning and the relative weight given to cognitive and social processes are key issues that serve to guide the specification and presentation of goals. In the section which follows, we begin by describing goals for mathematics education which are over-arching and expressed at a very general level. Then we describe two different approaches to thinking about skills and concepts. First we discuss Perry and Dockett's (2008) concept of *powerful mathematical ideas* as a unifying approach to emphasising both processes and content. Then we consider Sarama and Clements' (2009) goals, which they refer to as *big ideas* that they present as content-oriented goals, while stressing processes as implicit in these goals.

Specifying Goals

Overarching Goals for Mathematics Education

Higher-Order Thinking

Taking an international perspective, Cai and Howson (2013) argue that there are commonly accepted learning goals in school mathematics – the development of knowledge and skills along with an emerging emphasis on the development of higher-order thinking skills. In the absence of commonly accepted definitions, they utilise Resnick's (1987) characterisation of higher-order thinking as non-algorithmic, complex, with multiple solutions; involving nuanced judgement, application of multiple criteria, uncertainty, self-regulation, imposing meaning; and effortful. Cai and Howson (2013) draw attention to the flexibility and self-monitoring (meta-cognition) that these skills involve. These authors also emphasise the ability to work together with others as essential to the development of higher-order skills.

Specific mathematical processes employ higher-order skills. While some of these skills may appear to be very abstract in terms of young children's mathematical thinking, all of them have their genesis in early childhood. For instance, Australian researchers Perry and Dockett (2008) point out that argumentation is now seen as central to the mathematics development of young children. Citing Krummheuer (1995, p. 229), they define argumentation as a 'social phenomenon, when co-operating individuals [try to] adjust their intentions and interpretations by verbally presenting the rationale of their actions.' They are concerned that there is recognition of what argumentation might look like in young children.

Engaging with Powerful Mathematical Ideas

Taking a strongly process-oriented approach and from a vantage point of sociocultural theory, Perry and Dockett (2008) propose a list of powerful mathematical ideas, to which they believe most young children have access (see Table 4.1). Again, an example of a 'powerful idea' is argumentation. These ideas combine processes and content, with processes foregrounded. In their view, knowledge and skills are developed through engaging in mathematical processes. They identify four important issues for the development of knowledge and skills, particularly at the school level: models and modeling, language, technology, and assessment. In their judgement, these key processes, when well-conceived, understood and promoted by teachers, can serve as critical drivers in the development of the powerful mathematical ideas that children need to understand. They emphasise children's purposeful use of mathematics in their everyday lives in prior-to-school settings and in out-of-school settings. They focus on the centrality of using children's understandings built up through engagement in everyday activity as a basis for learning and teaching mathematics in the range of early education settings.

Exploring the Big Ideas in Mathematics Learning

Mathematics educators, especially in the US, make frequent reference to the need for teachers to understand the 'big ideas' in young children's mathematics learning and use them to connect ideas in mathematics (NCTM, 2000). Baroody, Purpura and Reid (2012, p. 164) explain that these ideas interconnect various concepts and procedures within a domain and across domains. They represent 'big leaps' in the development of children's reasoning and can be seen, according to Fosnot and Dolk (2001, p. 11), as both 'deeply connected to the structures of mathematics…[and] also characteristic of shifts in learners' reasoning'.

However, a definite list as to what exactly these ideas might be is more difficult to ascertain. Some examples of what different commentators understand as big ideas are to be found in the literature. For instance, enumeration (determining a set's numerical value) is underpinned by a set of mathematical ideas such as cardinality (e.g., Ginsburg, 2009b). Unitising underlies the understanding of place value (e.g., Fosnot & Dolk, 2001).

From an early childhood perspective, Sarama & Clements (2009) define their big ideas in mathematics as

> *overarching clusters and concepts and skills that are mathematically central and coherent, consistent with children's thinking, and generative of future learning. This organisation reflects the idea that children's early competencies are organised around several large conceptual domains.* (pp. 16–17)

These authors appear to conflate the idea of goals with that of 'big ideas' (e.g., Smith-Chant, 2010a). They suggest that in early childhood mathematics there are about twelve big ideas that need to be built up incrementally over time (see Table 4.1). They identify at least one accompanying big idea for verbal and object counting: counting can be used to find out how many are in a collection. Their big idea of composition and decomposition of shape has at least one associated big idea: geometric shapes can be described, analysed, transformed and composed and decomposed into other shapes. Sarama and Clements' work on learning trajectories (see *Chapter 5*) is built on the big ideas, or goals, they identify as essential for the learning and teaching of early mathematics. The explication of each of their goals is based on several decades of work in the cognitive sciences which they synthesised and presented in the form of developmental progressions. Clements and Sarama (2009a, p. 6) stress that their goals focus on far more than facts and ideas, and that processes and attitudes are important in each goal. However, processes are not explicit in their specification of goals. This issue of how processes are presented and integrated with skills and content is one that is critical in terms of the presentation of the redeveloped curriculum.

Table 4.1. Specifying Goals: Different Approaches

Perry and Dockett (2008)	Sarama and Clements (2009)	Primary School Mathematics Curriculum (1999)
Powerful mathematical ideas (a sociocultural perspective)	Big ideas (a cognitivist perspective)	A socioconstructivist/ sociocultural perspective
▪ Mathematization ▪ Connections ▪ Argumentation ▪ Number sense and mental computation ▪ Algebraic reasoning ▪ Spatial and geometric thinking ▪ Data and probability sense	▪ Counting ▪ Ordering numbers ▪ Recognising number and subitising ▪ Knowing different combinations of numbers ▪ Adding and subtracting ▪ Multiplying and dividing ▪ Measuring ▪ Recognising geometric shapes ▪ Composing geometric shapes ▪ Comparing geometric shapes ▪ Spatial sense and motions ▪ Patterning and early algebra	▪ Applying and problem-solving ▪ Understanding and recalling ▪ Communicating and expressing ▪ Integrating and connecting ▪ Reasoning ▪ Implementing ▪ Early mathematical activities ▪ Number ▪ Algebra ▪ Shape and space ▪ Measures ▪ Data

The Structure of Curriculum Materials

A particular issue in relation to curriculum implementation has been a widely acknowledged difficulty in the integration of processes, skills and content, with teachers placing greater emphasis on procedural aspects of mathematics than on broader educational goals (Anderson, White, & Sullivan, 2005; Eivers et al., 2010; Handal & Herrington, 2003; Ross, McDougall, Hogaboam-Gray, & LeSage, 2003). Numerous factors have been identified to explain the mismatch between 'intended', 'enacted' and 'attained' curricula (Cuban, 1993; Robitaille & Garden, 1989) – most particularly teacher beliefs (Anderson et al., 2005; Handal & Herrington, 2003). However, some attention has been given recently to the objective structure of curriculum materials (Herbel-Eisenmann, 2007). While acknowledging the complex and multifaceted nature of the teacher-curriculum relationship, Remillard (2005) urges curriculum developers to take account of this relationship in the design of materials:

> *…[C]urriculum materials have a number of characteristics beyond the specific mathematical content and pedagogy represented in the text. These characteristics include the look and voice of the text and its subjective scheme or how it is perceived. It is critical that curriculum developers pay careful attention to the multiple ways that their materials communicate with the teacher. They must consider how they are addressing the teacher through the design of their materials, how they expect the teacher to respond to their suggestions, and how they represent what it means to use their resource.* (p. 240)

Comparing the Perry and Dockett specification of goals with the Sarama and Clements specification, we can see that, while the former foregrounds processes but includes content areas, the latter appears to focus on content and sees processes as implicit. The question is whether one or the other approach is preferable in terms of key organisers in the redeveloped maths curriculum. There is also the issue of which presentation best promotes continuity of experiences and pedagogy in different settings. Advantages and disadvantages can be identified with both approaches.

The Perry and Dockett specification foregrounds processes. This is consistent with their sociocultural perspective on learning. They lead with mathematization, a process which can actually be seen as content since as children explore a mathematical idea they are involved in the content of mathematics (e.g., Fosnot & Dolk, 2001). There are two readily identifiable arguments for a specification such as this one. The first relates to coherence – among the conditions that Schoenfeld (2002) identifies for high quality mathematics teaching is the development of 'coherent curricula rather than disconnected sets of activities' (p. 9). Given the sociocultural/situative view of mathematics, of mathematics education and of pedagogy espoused in previous chapters of this report, a specification with a strong focus on process makes for a greater degree of coherence. The second relates to how the curriculum presents to teachers. The Perry and Dockett list is a balanced one with processes listed before content, thus signalling to educators a revised emphasis.

In Ireland, the PSMC is presented in two distinct sections. In many respects, this curriculum is quite detailed. It includes a skills development section which describes the skills that children should acquire as they develop mathematically. It also includes a number of strands which outline content that is to be included in the mathematics programme at each level. Each strand includes a number of strand units. These are further broken down, mainly with reference content objectives, with a number of these related to each strand unit. However, research now suggests an alternative approach to breaking down the goals into large numbers of objectives. This involves a specification of key mathematical ideas and critical transitions.

Breaking Down the Goals: Critical Transitions within Mathematical Domains

Goals for mathematics learning can be developed at different levels of detail. Above we saw that Sarama and Clements (2009) appear to conflate the idea of goals with that of 'big ideas'. However they also allude to accompanying ideas which indicate key insights in relation to children's understanding of the goal or big idea. For instance they identify the notion that counting can be used to identify how many are in a collection, as a key insight in relation to verbal and object counting. Simon and Tzur (2004) also reference cardinality, but refer to it as a key developmental understanding (KDU). The big leaps and shifts in reasoning as described by Fosnot and Dolk (2001) and referenced earlier in the chapter appear to us to be analogous to KDUs.

Simon considers that KDUs are essential in that they identify 'critical transitions that are essential for children's understanding of a particular concept or domain' (p. 360). Furthermore, he argues that they provide the basis for the specification of what he terms important learning goals (along a developmental progression). From a cognitive science perspective, the identification of critical transitions and their incorporation into the curriculum as goal statements is essential, since doing so allows for progressive conceptual development, from key conceptual foundations to the incremental construction of understanding. For example, children need to learn about units of quantification (Sophian, 2004) in ways that allow them to easily build on this knowledge as they meet new (key) ideas and as their concepts about these develop. While not previously made explicit as KDUs, Simon points to important examples of these in the literature. Amongst other work referenced by him in this respect is the work of Gelman and Gallistel (1978), Piaget (1952), and Steffe and Cobb (1988), all in the area of number. He sees their work as clearly identifying KDUs (cardinality, composite units and conservation of number) that are central to children's abilities to conceive of and work with number. Essentially what Sarama and Clements have done is to extract these from the literature and use them to build developmental progressions for their big ideas in mathematics.

Some efforts to identify key elements of domain-related content are to be found in the literature. In the United States the *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (NCTM, 2006) identified what they considered to be the important ideas and major themes which should receive special attention at particular points in time, across the domains of Number, Geometry and Measure. The aim of the focal points is to show teachers how they might build on important mathematical content and connections identified for each grade level (p. 3). The *Focal Points* approach 'focuses on a small number of significant mathematical targets for each grade level…[and] the most significant mathematical concepts and skills…' (p. 1). They are presented in narrative rather than list format, and describe the content emphases for different grade levels. Table 4.2 is an example of the kindergarten curriculum focal points.

Table 4.2. Kindergarten Curriculum Focal Points

Kindergarten Curriculum Focal Points	Connections to the Focal Points
Number and Operations: Representing, comparing, and ordering whole numbers and joining and separating sets. Children use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set, creating a set with a given number of objects, comparing and ordering sets or numerals by using both cardinal and ordinal meanings, and modeling simple joining and separating situations with objects. They choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognising the number in a small set, counting and producing sets of given sizes, counting the number in combined sets, and counting backwards.	*Data Analysis*: Children sort objects and use one or more attributes to solve problems. For example, they might sort solids that roll easily from those that do not. Or they might collect data and use counting 'to answer such questions as, 'What is our favourite snack?' They re-sort objects by using new attributes (e.g., after sorting solids according to which ones roll, they might re-sort the solids according to which ones stack easily). *Geometry*: Children integrate their understandings of geometry, measurement, and number. For example, they understand, discuss, and create simple navigational directions (e.g., 'Walk forward 10 steps, turn right, and walk forward 5 steps').

Table 4.2. Kindergarten Curriculum Focal Points (*continued*)

Kindergarten Curriculum Focal Points	Connections to the Focal Points
Geometry: Describing shapes and spaces Children interpret the physical world with geometric ideas (e.g., shape, orientation, spatial relations) and describe it with corresponding vocabulary. They identify, name and describe a variety of shapes, such as squares, triangles, circles, rectangles, (regular) hexagons, and (isosceles) trapezoids presented in a variety of ways (e.g., with different sizes or orientations), as well as such three-dimensional shapes as spheres, cubes, and cylinders. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.	*Algebra*: Children identify, duplicate and extend simple number patterns and sequential and growing patterns (e.g., patterns made with shapes) as preparation for creating rules that describe relationships.
Measurement: Ordering objects by measureable attributes Children use measureable attributes, such as length or weight, to solve problems by comparing and ordering objects. They compare the lengths of two objects both directly (by comparing them with each other) and indirectly (by comparing both with a third object), and they order several objects according to length.	

Taken from *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics: A Quest for Coherence*, by National Council of Teachers of Mathematics (NCTM) (2006), p. 12. Reston, Virginia: NCTM.

In Table 4.2. Kindergarten Curriculum Focal Points we see that the domains of Number, Geometry and Measure outline the key ideas within each domain. The key ideas are broken down into what appear to be critical transitions. These are framed as learning outcomes. It seems to us that the *Focal Points* approach provides a basis for structuring the curriculum at content level with the content-level descriptors providing a basis for identifying learning outcomes. We return to this topic in Report No. 18 (Chapter 3, Section: *Content Areas and Curriculum Presentation*).

Conclusion

The goals of a curriculum must be aligned with its underlying theory. A sociocultural stance implies that goals must be consistent with the view of learning as a socially and culturally embedded process which takes place in interaction with others. A curriculum which identifies goals and breaks them down into key mathematical ideas and critical transitions can help educators to move towards more focused teaching and assessment approaches.

The key messages presented in this chapter are as follows:

- Curriculum goals should reflect new emphases on ways to develop children's mathematical understandings, and to foster their identities as mathematicians. In the redeveloped curriculum both processes and content should be clearly articulated as goals.
- The approach whereby mathematical processes are foregrounded but content areas are also specified is consistent with a participatory approach to mathematics learning and development.
- General goals need to be broken down for planning, teaching and assessment purposes. Critical ideas derived in this way indicate the shifts in mathematical reasoning required for the development of mathematical concepts. An understanding of mathematical development enables teachers to provide support for children's progression towards curriculum goals.

These issues are addressed in *Chapter 5* and *Chapter 6*, and we return to them again in Report No. 18 (Chapter 3, Section: *Content Areas and Curriculum Presentation*). In *Chapter 5* we discuss different approaches to the specification of learning paths and teaching paths, designed to enable learners to progress towards the goals of the curriculum.

CHAPTER 5

The Development of Children's Mathematical Thinking

A Historical Perspective

The idea of stages of development in children's mathematical thinking and learning is most often associated with Piaget. His theory identified a sequence of what he considered to be invariant stages through which children's thinking progresses – from sensorimotor to pre-operational to concrete operational and finally formal operational. Each stage was characterised by a particular type of thinking applicable across many domains. But we now know that development is not equal across mathematical domains; for instance, children may conserve number before they can conserve mass or capacity (e.g., Ryan & Williams, 2007). Also, within domains, development is gradual rather than step-like (Casey, 2009). We know that the context, the materials, the task and especially the language used can make a difference to how children reason when faced with any mathematical task (e.g., Donaldson, 1984; NRC, 2005). Research also shows that contrary to Piaget's proposition, there is no clear progression from concrete to abstract thinking in children's development (e.g., NRC, 2009). Young children's thinking is both concrete and abstract (e.g., Ginsburg, 2009a).

One framework for mathematics learning and teaching that is receiving attention in countries as diverse as Japan, Korea, Australia, as well as in Europe and the United States is that of learning trajectories, also sometimes referred to as learning paths (e.g., Bobis et al., 2005; Daro et al., 2011; Griffin, 2004; Lewis & Tsuchida, 1998; Stigler & Thompson, 2012; van den Heuvel-Panhuizen, 2008). Interest in learning trajectories/learning paths is not confined to mathematics. They are also being developed in science and in literacy (e.g., Daro et al., 2011). The history of learning trajectories in mathematics education can be traced at least as far back as the work of Treffers (1987), whose perspective was that of the RME school (see below). The work of Simon (1995) was an important catalyst which resulted in intense interest in his (social constructivist) articulation of the concept of hypothetical learning trajectories (HLT). Work by American researchers Sarama and Clements is also included here since it currently features prominently in early childhood mathematics education, especially in the United States.

In this chapter we explore the progression from the Piagetian idea of stages of development to the idea of learning trajectories and learning paths. We see how, through the 1970s and 1980s, the idea of levels of mathematical thinking was developed as a concept of interest for Realistic Maths Education (RME) theorists. Then in the 1990s the concept of hypothetical learning trajectories (HLTs) was advanced by Simon. He saw HLTs as key elements in mathematics teaching cycles. More recently, in the United States, Sarama and Clements have developed their learning trajectories for learning and teaching early mathematics (e.g. 2009). Each of these developments is of interest in the context of the current review, and potentially informative in relation to issues of curriculum, assessment, equity and teacher education.

From Stages of Development to Levels of Sophistication in Thinking

In recent decades cognitive scientists have focused on knowledge construction and on the thinking that children use to solve problems. This concerns children's internal cognitive structures and processes and researchers' interpretations and understandings of what is happening in relation to the child's thinking (Cobb, 2007). Piaget's theory has been adapted to gain insights into children's mathematical thinking and how that thinking changes and develops over time. Interests are focused on how change occurs, most significantly qualitative changes in children's mathematical reasoning (e.g., Casey, 2009). Both constructions of meaning for specific mathematics topics and the characterisation of children's developing conceptualisation and reasoning in terms of different levels of sophistication in thinking are important emphases (Battista, 2004, p. 186).

Developing Children's Mathematical Thinking: Three Approaches

An emphasis on helping learners to move through increasingly sophisticated levels of mathematical reasoning and understanding is now seen as a key focus for mathematics education from a cognitive science point of view (e.g., NRC, 2009). Gravemeijer (2004) argues that a pedagogy which supports this is generally well-articulated, i.e., it is 'elaborated in terms of classroom culture, social norms, mathematical discourse, mathematical community, and a stress on inquiry and problematizing' (p. 106). However, he argues that it is necessary to draw attention to the curriculum counterpart of this innovative pedagogy. He points out that in the 1960s and 70s curriculum design took as its starting point the knowledge and expertise of experts in order to construct learning hierarchies. The problem with that approach was that it did not take into account the perspective and personal input of the learner. Proposed revisions to the mathematics curriculum will need to consider how to ensure that this issue is addressed, particularly in guidance on pedagogy.

Below, we present three different approaches to helping teachers in the task of developing children's mathematical thinking in the way described above. What they have in common is the fact that each subscribes to the idea of learning trajectories or learning paths. Where they diverge is in the roles they see these playing in the teaching/learning process.

The First Approach: Working with Children's Thinking and Understanding (RME)

The notion of levels of thinking was first advanced by Freudenthal who drew, in particular, on the work of Pierre and Dina van Hiele. They were his students, and they had developed a model of geometric thinking at the University of Utrecht, Netherlands in 1957 (Crowley, 1987). The basis of this model is that thinking develops from an initial visual level through increasingly sophisticated levels, that is, analysis, abstraction, deduction and rigour.

Freudenthal (1971) expanded on this model in his theory on the learning of mathematics:

> *The van Hiele levels of the learning process are often characterised by a logical feature: the activity on one level is subjected to analysis in the next, the operational matter on one level becomes a subject matter on the next level.* (p. 417)

This means that mathematical activities that have been carried out in an informal way initially later become more formal as a result of reflection (this is an aspect of mathematization as described by RME theorists e.g., van den Heuvel-Panhuizen, 2003). Early mathematics is constituent of and not separate from formal mathematics, implying that RME ideas about levels of thinking and their implications for pedagogy are elaborations of children's earlier understandings.

Key Features

The RME approach entails directing teachers' attention to children's understandings of mathematics and engaging children with rich problem contexts. Instruction evolves to suit the learners. When first introduced in the 1970s, this was a novel way to approach teaching. A feature of the approach is that children work with realistic problems. These allow them to imagine. The problems can include contexts from real-world situations, but also problems from the fantasy world of fairy tales or from the formal world of mathematics (van den Heuvel-Panhuizen, 2003). A second essential feature is the use of models developed by the children as a basis for teaching and learning. These have a specific role in that they provide the context in which children can be supported in the activity of mathematizing, i.e., *'the analysing of real world problems in a mathematical way'* (Treffers & Beishuizen, 1999, p. 32). A third feature is that different levels of understanding can be distinguished and as children pass through these levels, models can have an important role in level-raising: they are seen as bridges between informal understanding and the abstraction of formal ideas. A model can, for instance, include materials, visual sketches or symbols. Models share two important characteristics: they have to be rooted in realistic contexts and they must be flexible and applicable on a more general level. Models can be models of a situation initially, but then they must be capable of becoming models for organising new problems and reasoning about these in a mathematical way (van den Heuvel-Panhuizen, 2003). The models are formulated by children themselves in the course of their engagement with the problem and they gradually gain a better understanding of a rich, meaningful problem situation by describing and analysing it with more and more advanced means. By going through a series of modeling cycles, they finally develop an effective model with which they can also take on other (similar) complex problem situations (p. 29). See Report No. 18, Chapter 2, Section: *Emphasis on Mathematical Modeling*.

The Teacher's Task

The RME position is that levels of thinking or understanding can be specified in a general sense and it is the teacher's task to explore children's understandings at the different levels and use these to progress learning.

While general hypothetical learning trajectories are used as the basis of the teacher's work these are seen as initial starting points which are subject to constant revision by the teacher as a learning trajectory specific to his or her particular classroom emerges. From this perspective, teachers learn to use learning-teaching trajectories that fit their particular situation (Gravemeijer, 2004). The trajectory provides an overview of levels of understanding in a domain. It should not be seen in a linear way: there can be variations in the steps. The trajectory sets out important signposts, and allows teachers to discern the differences in children's understandings. This approach is very much about developing the teacher's abilities to make decisions about how best to help children with 'intermediate attainment targets', on the way to achieving general goals. These are seen as the crucial steps or 'landmarks towards which the learning can be oriented' (van den Heuvel-Panhuizen, 2008, p. 9).

Initial work in their project of developing learning-teaching trajectories has focused on the domain of number since it is seen as an area of concern for teachers and a good place to begin. This work is shown in Table 5.1 below. Some work has also been done on Geometry and Measures (van den Heuvel-Panhuizen & Buys, 2008). See also Report No. 18, Chapter 3, Sections: *Measurement*; *Geometry and Spatial Thinking*.

Table 5.1: A Learning-Teaching Trajectory for Number

Emergent numeracy (preschool)	
Elements	
▪ Recognising 'two-ness', 'three-ness', and 'many-ness' as a property of a group of objects ▪ Learning to recall the number sequence ▪ Imitating resultative counting ▪ Symbolising by using fingers	
Growing number sense (K1 and K2)	
Elements	**Levels**
▪ Learning to count	Children know the counting sequence, at least up to 10.
▪ Learning to count and calculate	Within what are for them meaningful context situations, children are able to count to at least 10, arrange numbers in the correct order, make reasonable estimates, and compare quantities being more, less or equal (level 1).
▪ Context bound counting and calculating	Children can order, compare, estimate and count up to 10 objects. They are also able to select a suitable strategy for simple addition or subtraction situations in such things as concealment games for up to 10 objects (level 2).
▪ Towards pure counting-and calculating via symbolisation	Children can represent physical numbers up to 10 on their fingers and with lines and dots, and are able to use these skills for 'adding up' and 'taking away'.
Calculations up to 20 (G1 and G2)	
Elements	**Levels**
▪ Calculations by counting, supported where necessary by counting materials ▪ Non-counting based calculating by structuring with the help of suitable models	The children can recite the number sequence up to 20 and can count up and down from any number in this domain. They can also put numbers up to 20 into context by giving them a real world meaning, can structure then by doubling and using groups of five and 10, and place them on an empty number line from 0 to 20.
▪ Formal calculation using numbers as mental objects for smart and flexible calculation without the need for structured materials	The children should be able to add and subtract quickly, in the number area up to 20 by structuring the numbers and, in time, they should be able to perform formal calculations with the help of remembered number properties. They should also be able to use this skill in elementary context situations and be able to both understand and use some conventional mathematical notation.

Adapted from van den Heuvel-Panhuizen, 2008

By way of defining a learning-teaching trajectory, van den Heuvel-Panhuizen (2008, p. 13) states that there are three interwoven meanings:

- a learning trajectory that gives a general overview of the learning process of the students
- a teaching trajectory, consisting of didactical indications that describe how the teaching can most effectively link up with and stimulate the learning process
- a subject matter outline, indicating which of the core elements of the mathematics curriculum should be taught.

van den Heuvel-Panhuizen (2008) describes how the learning-teaching trajectory, or TAL[8], with intermediate targets for calculation with whole numbers in primary schools builds on children's earlier numerical experiences. They present TALs for number for the youngest children at three levels. They call the first level the level of 'Emergent numeracy' (preschool years), the second the level of 'Growing number sense '(kindergarten 1 and 2), and the third 'Calculations up to 20' (grades 1 and 2). Further discussion of the relevance of this work as it applies to Number, Geometry and Measures is presented in Report No. 18 (Chapter 3, Section: *Content Areas*).

The intention is to extend this work into secondary education. The learning-teaching trajectory is seen as part of 'the longitudinal perspective' (p. 11) that all teachers need to hold. It is seen to go beyond a textbook and beyond tests, but to focus on the attainment targets and a general indication of teaching activities that can contribute to achieving these. In particular, domain-specific 'levels' of understandings are seen as potentially useful for specifying communal trajectories, i.e., ones that apply to particular school years or grades. They are also seen as useful in relation to level-raising, i.e., moving children towards the final core goals of mathematics education at primary level. In addition, it is suggested that TALs provide a means whereby teachers can monitor children's development.

The Second Approach: Teacher-Generated Hypothetical Learning Trajectories (Simon)

Working with the tension created by the need to attend to predetermined goals for children's learning whilst at the same time being responsive to children's thinking, Simon (1995) developed a theoretical model of teacher decision-making with respect to mathematics tasks. Simon-proposed HLTs comprise the learning goal, the learning activities and a description of the thinking and learning that students might engage in.

8 In Dutch, learning-teaching trajectories are referred to as TALs. TAL stands for Tussendoelen Annex Leerlijinen, translated into English as *intermediate attainment targets in learning-teaching trajectories* (Fox, 2005/2006).

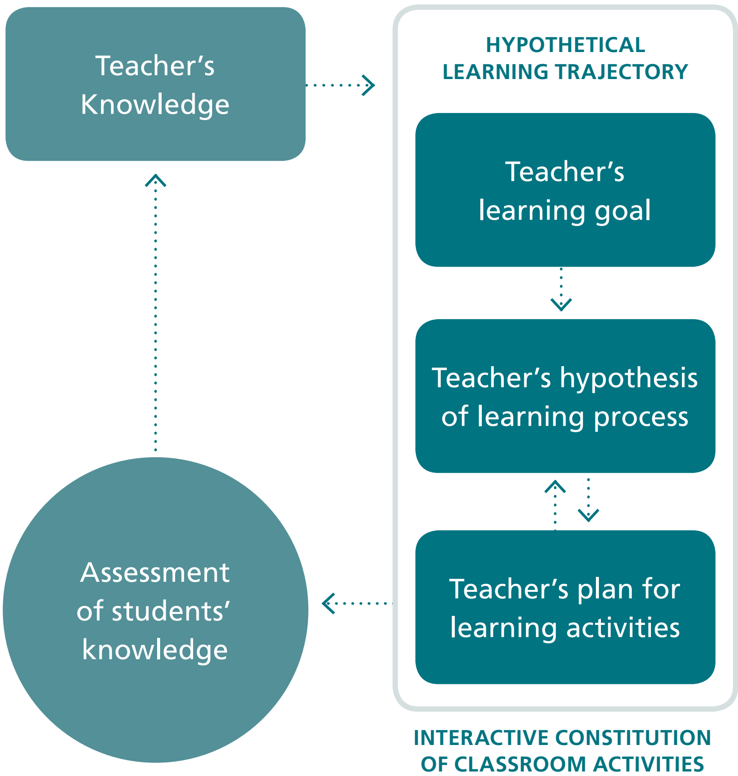

Figure 5.1. Adapted from Reconstructing Mathematics Pedagogy From a Constructivist Perspective by M. Simon, 1995. *Journal for Research in Mathematics Education*, 26(2), p. 136.

He emphasised the unpredictable nature of teaching mathematics, and the need for continuous modifications to the teaching plan. Simon (1995) describes a dynamic cycle wherein:

> *as students begin to engage in the planned activities, the teacher communicates and observes the students, which leads the teacher to new understandings of the students' conceptions. The learning environment evolves as a result of interaction amongst the teacher and students as they engage in the mathematical content…it is what the students make of the task and their experience with it that determines the potential for learning.* (p. 133)

Simon's explication of the term hypothetical learning trajectory emphasises the teacher's prediction as to the path by which learning might proceed. It emphasises its hypothetical nature – the actual learning trajectory is not known in advance. For Simon, learning trajectories are essentially a teaching construct. This is similar to the RME perspective. While Simon's (1995) original articulation of the HLT did not specify teaching activities, he did later address the issue of tasks (Simon & Tzur, 2004). His concern then was that tasks would not be left to intuition or trial and error but would be deliberately constructed to promote the learning process.

The teacher, in designing a learning trajectory, must consider both the tasks to be used and the learning goals. With respect to the conceptual learning goals, Simon (2006) proposes the identification of key developmental understandings (KDUs):

> *significant landmarks in students' mathematical development…understandings that account for differences between those learners who show evidence of more sophisticated conceptions from those who exhibit less sophisticated conceptions.* (p. 370)

The significance of Simon's approach is the influence it had on subsequent work on ways of developing children's mathematical thinking. One example of this is the body of work on learning trajectories developed by Sarama and Clements.

The Third Approach: Pre-Specified Developmental Progressions as a Basis for Learning Trajectories (Sarama and Clements)

In the United States, Sarama and Clements synthesised relevant research from cognitive science on the learning of mathematics from birth to age 8. From this synthesis they developed their learning trajectories (e.g., 2009). Table 5.2 shows Sarama and Clements' learning trajectory for volume measurement. We draw the reader's attention to the level of detail presented in these developmental progressions and the accompanying hypothesised mental actions that appear in the third column. Note also the age column on the left. The authors state that these are age-indicators based on research and are provided only as a general guide.

Table 5.2. A Developmental Progression for Volume Measurement

Age (Years)	Development Progression	Actions on Objects
0–3	*Volume/Capacity:* **Volume Quantity Recognizer** Identifies capacity or volume as attribute. ▪ Says, 'This box holds a lot of blocks!'	Perceives space and objects within the space.
4	**Capacity Direct Comparer** Can compare two containers. ▪ Pours one container into another to see which holds more.	Using perceptual objects, internal bootstrap competencies to compare linear extent (see the length trajectory for 'Direct Comparer') or recognize 'overflow' as indicating the container 'poured from' contains more than that 'poured into.'
5	**Capacity Indirect Comparer** Can compare two containers using a third container and transitive reasoning. ▪ Pours one container into two others, concluding that one holds less because it overflows, and the other is not fully filled.	A mental image of a particular amount of material ('stuff') can be built, maintained, and manipulated. With the immediate perceptual support of the containers and material, such images can be compared. For some, explicit transitive reasoning may be applied to the images or their symbolic representations (i.e., object names).
6	*Volume/Spatial Structuring:* **Primitive 3-D Array Counter** Partial understanding of cubes as filling a space. ▪ Initially, may count the *faces* of a cube building, possibly double-counting cubes at the corners and usually not counting internal cubes. ▪ Eventually counts one cube at a time in carefully structured and guided contexts, such as packing a small box with cubes.	With perceptual support, can visualize that 3-D space can be filled with objects (e.g., cubes). With strong guidance and perceptual support from pre-structured materials, can direct the filling of that space and recognize that filling as complete, but often only intuitively. Implicit visual patterning and constraints of physical materials guides placement of cubes.

Table 5.2. A Developmental Progression for Volume Measurement (*continued*)

Age (Years)	Development Progression	Actions on Objects
7	**Capacity Relater and Repeater** Uses simple units to fill containers, with accurate counting. ▪ Fills a container by repeatedly filling a unit and counting how many. ▪ With teaching, understands that fewer larger than smaller objects of units will be needed to fill a given container.	See the learning trajectory level, Length Unit Relater and Repeater.
7	*Volume/Spatial Structuring:* **Partial 3-D Structurer** Understands cubes as filling a space but does not use layers or multiplicative thinking. Moves to more accurate counting strategies e.g.: ▪ Counts unsystematically, but attempts to account for internal cubes. ▪ Counts systemically, trying to account for outside and inside cubes. ▪ Counts the numbers of cubes in one row or column of a 3-D structure and using skip counting to get the total.	Builds, maintains, and manipulates *mental* images of composite shapes, structuring them as composites of individual shapes and as a single entity – a row (a unit of units), then a layer (a 'column of rows' or unit of unit of units). Applies this composite unit repeatedly, but not necessarily exhaustively, as its application remains guided by intuition.
8	*Area/Spatial Structuring:* **3-D Row and Column Structurer** ▪ Counts or computes (row by column) the number of cubes in one row, and then uses addition or skip counting to determine the total. ▪ Computes (row times column) the number of cubes in one row, and then multiplies by the number of layers to determine the total.	Builds, maintains, and manipulates *mental* images of composite shapes, structuring them as composites of individual shapes and as a single entity – a layer (a unit of unit of units) of *congruent* cubes. Applies this composite unit repeatedly *and exhaustively* to fill the 3-D array – coordinating this movement in 1–1 correspondence with the elements of the orthogonal column. If in a measurement context, applies the concept that the length of a line specifies the number of unit lengths that will fit along that line. May apply a skip counting scheme to determine the volume.

Taken from *Early Childhood Mathematics Education Research: Learning Trajectories for Young Children* by J. Sarama and D. Clements, 2009, pp. 306–607. New York, NY: Routledge.

The key difference between their work and that of either the RME school, or Simon, is the emphasis they place on developmental progressions. These are learning paths 'through which children move through levels of thinking' (2009, p. 17).

Clements and Sarama (2004) set out to emphasise both learning processes and teaching processes together:

> *We conceptualise learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesised to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain.* (p. 83)

While initially they used the term hypothetical learning trajectories, more recently they tend to use the term learning trajectories while still maintaining that their trajectories are hypothetical. As discussed earlier, Sarama and Clements see their goals as the twelve big ideas they identify for early mathematics (see Chapter 4, *Table 4.1. Specifying Goals: Different Approaches*). The research-based developmental progressions or learning paths identify the levels of thinking that children progress through as they work towards the goal. These levels of thinking are at the core of the trajectory. The instructional tasks or teaching paths consist of 'sets of instructional tasks, matched to each of the levels of thinking in the progressions' (Clements & Sarama, 2009a, p. 2). Also, as noted above, age estimates are also provided as a general guide to when children might develop certain understandings.

The levels of thinking, as characterised by Clements and Sarama (2009b), are understood to be domain-specific:

> *Children are identified to be at a level when most of their behaviours reflect the thinking-ideas and skills of that level…Levels are not absolute stages. They are benchmarks of complex growth that represent distinct ways of thinking…sequences of different patterns of thinking and reasoning. Children are continually learning, within levels and moving between them…Children may also learn deeply and jump ahead several 'levels' in some cases.* (p. 5)

Comparing the Three Approaches

Definitions and Characteristics

The learning trajectory concept is interpreted, re-presented and applied in a range of different ways. Table 5.3 presents the definitions of learning trajectories for each of the three approaches described above. We see these definitions as indicative of some of the subtle nuances and differences inherent in each of the approaches.

Table 5.3. Three Approaches to Defining Learning Trajectories

Realistic Maths Education Trajectories	Simon's Hypothetical Learning Trajectories	Sarama and Clements' Trajectories
A learning-teaching trajectory has three interwoven meanings: ▪ a learning trajectory that gives a general overview of the learning process of the students ▪ a teaching trajectory, consisting of didactical indications that describe how the teaching can most effectively link up with and stimulate the learning process ▪ a subject matter outline, indicating which of the core elements of the mathematics curriculum should be taught.	A hypothetical learning trajectory is composed of the learning goal, the learning activities and a description of the thinking and learning that students might engage in.	Learning trajectories are descriptions of children's thinking and learning in a specific mathematical domain and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesised to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain (Clements & Sarama, 2004, p. 83).

Learning-teaching paths from the RME perspective have much in common with sociocultural/situative perspectives. For example, establishing an appropriate classroom culture for successful learning in mathematics is emphasised. Discussion is the context within which the teacher focuses on what Gravemeijer (2004) refers to as 'the inventions of the students' (p. 126). The approach used by Simon and by Sarama and Clements both take a cognitive science approach to promoting children's mathematical understanding. Simon's theoretical notion of HLTs was important in that it moved away from any notion of learning progressing in a linear way. It recognised that not all children follow the same path towards understanding. It sought to apply constructivist theory to teaching approaches. Similarly, Fosnot and Dolk (2001) describe what they call 'a landscape of learning' and how children traverse this as they engage in mathematics in the classroom – 'They go off in many directions as they explore, struggle to understand, and make sense of their world mathematically' (p. 18).

Fosnot and Dolk describe how children's learning paths twist and turn, cross each other and often use an indirect route to get a particular landmark. They illustrate this in Figure 5.2 below.

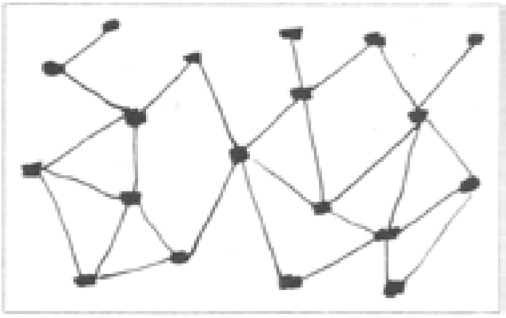

Figure 5.2: Reprinted with permission from Young Mathematicians at Work: Constructing Number Sense, Addition and Subtraction by C. T. Fosnot & M. Dolk, 2001, p. 18. Portsmouth, NH: Heinemann. All rights reserved.

In contrast, because the detailed learning trajectories developed by Sarama and Clements are presented linearly, educators may incorrectly infer that mathematical development is linear.

It seems to us that one of the confounding issues for readers in dealing with the dense and complex literature surrounding the various learning trajectories and learning paths is that, while they are referred to as learning trajectories, their central purpose is generally as a pedagogical tool. Simon and Tzur (2004) clearly state that a HLT is a vehicle for planning the learning of concepts, while Clements and Sarama (2004) consider Simon's HLTs as a way of describing the pedagogical thinking involved in teaching mathematics for understanding.

The different ways in which the trajectories model has been developed is perhaps also a response to the context in which individual theorists see the trajectories being used. For instance, in the US there is a great deal of concern to accelerate both the professional development of teachers of children aged 3–6 years, and to specify standards for early childhood mathematics education. The learning trajectory approach is seen as a way to meet both concerns (e.g., NRC, 2009). On the other hand, RME perspectives appear to focus more on working in a contingent way with children's ideas. Theorists from this perspective generally see learning trajectories, or TALs, as a tool to be used in contexts characterised by teacher judgement, and where teaching is characterised by an emergent, creative and adaptive pedagogy focused on real problems located in children's own experiences. In Japan, for over a decade, learning trajectories have provided the basis for lesson study, i.e., detailed planning of research lessons by teachers. These lessons are taught, reflected on, analysed and redesigned by teachers and in this way instruction improves (Lewis & Tsuchida, 1998). See Report No. 18, Chapter 6, Section: *Frameworks for Thinking about Pedagogy; Using Tools for Teacher Preparation*.

Recognising Diverse Routes in Learning

From a sociocultural point of view, there are many possible routes that children may take to reach a common goal. Socioculturists emphasise the role that experiences and contexts play in determining what children learn, but also the role of context in determining what learning children might display during observations and assessments focused on ascertaining the extent of their understanding. Sarama and Clements (2009) describe how, in constructing their developmental progressions, they used the available research but, where none was available, they used judgement and best guesses to suggest a hypothetical path. Learning trajectories then can be regarded as 'invented cultural artefacts' that have been constructed in order to 'help students get from point A to point B' (Stigler & Thompson, 2012, p. 192). Taking a similar stance, Wager and Carpenter (2012) remind us that learning trajectories are cognitive constructs based on certain assumptions about the cognitive nature of knowledge '… they do not fully account for the situated nature of children's learning…they should be used in a way that considers and connects to children's experiences' (p. 198). Another observation suggests that learning trajectories based on tightly specified developmental progressions appear to have lost Simon's original focus on children's learning as it might unfold in interactions with the teacher and the accompanying decision-making that the teacher might engage in (Empson, 2011). In contrast, the TALs are much less detailed and thus explicitly suggest that development can follow different paths.

Our reflections suggest that any presentation of learning trajectories to educators would need to be couched in terms of their potential as reference tools and not as roadmaps.

Recognising Developmental Variation

The different ways in which the learning trajectories have been developed are to some extent a consequence of the perspectives of the theorists and the extent to which they subscribe to different social and/or cognitive perspectives. The idea of universal development is deeply ingrained in cognitive science and the idea that many children may do things differently during the course of their mathematical learning and development is a relatively new expectation influenced by sociocultural perspectives. The following joint statement from the US *National Association for the Education of Young Children/National Council for Teachers of Mathematics* (2002/2010) is helpful in stating a balanced position:

> *the research base for sketching a picture of children's mathematical development varies considerably from one area of mathematics to another. Outlining a learning path does not mean that we can predict with confidence where a child of a given age will be in that sequence. Developmental variation is the norm, not the exception. However children do tend to follow similar sequences, or learning paths, as they develop.* (p. 19)

We have seen, however, that some frameworks use age-related steps or indicators in order to present learning progressions. We see inherent dangers with this approach. The key point we wish to emphasise here is that the linking of stages of development with age-levels is problematic. Different children develop at different rates and their learning is strongly influenced by culture and experience. From our perspective, it is more theoretically coherent to conceive of development and learning as proceeding along a path which has significant markers. The learning trajectory or path for an individual child cannot be known in advance. In other words, any proposed trajectory is a hypothetical one. Hypothetical learning trajectories are not the same thing as preferred teaching trajectories or paths. We have seen that, according to the original construct, hypothetical learning trajectories are a teaching construct which are speculative as regards how a particular child may develop. On the other hand, preferred teaching trajectories or paths could be considered a useful framework of reference for planning on the part of teachers.

Curriculum Development and the Role of Learning Trajectories

Steffe (2004) raises the question, 'Whose job is it to design learning trajectories?'. First and foremost, responses to this question are reflective of a view of knowledge, of learning and of teaching. They also reflect understandings about teachers and teaching, and of autonomy and agency in relation to the profession. They relate to issues of teacher preparation and preparedness in working with learning trajectories, in conceptualising children's mathematical learning, in planning effectively, and in establishing an appropriate classroom culture for successful learning in mathematics.

Supporting Teachers in Planning

The mathematics curriculum is concerned with emphasising tasks that enable children to work in different ways, to organise and interpret tasks in ways that make sense to them while making use of different mathematical strategies. This necessitates the design of HLTs. Designing these is not an easy task. The teacher must understand children's mathematical conceptions and engage in conceptual analysis (e.g., Simon & Tzur, 2004, Clements & Sarama, 2004). In recognition of the need to support teachers in this regard, Gravemeijer (2004) proposed that they be offered 'a framework for reference and a set of exemplary instructional activities that can be used as a source of inspiration' (p. 107). Ready-made instructional sequences are rejected because the teacher will continually have to adapt to the actual thinking and learning of his or her students. The emphasis is on the local nature of the planning. The trajectories are developed in response to the children's ideas and follow the cyclical process outlined by Simon (1995) and described above. Clements and Sarama (e.g., 2009b) appear to take a somewhat different approach, one where much of the decision-making is done by mathematics educators and presented to teachers in the form of detailed specifications of teaching paths. It seems to us that mathematics education theorists, in dealing with this quandary of teachers' understanding and their generation of learning trajectories, have taken diverging approaches. The issue is really about the detailed specification of what Clements and Sarama (2004) describe as 'natural developmental progressions' (p. 83). While these authors and others coming from a mainly cognitive science perspective see such specification as unproblematic, theorists coming from a sociocultural or similar perspective (for example, RME) are likely to temper such a position in favour of an approach which emphasises more explicitly the hypothetical nature of learning paths.

While conceding that detailed developmental sequences are most likely over-simplified descriptions of development, we see them as having a role in terms of assessment. They can provide a theoretical framework for guiding teacher judgements (e.g., Ginsburg, 2009b). Their strength here lies in the fine-grained analysis of learning that they provide. They can serve as reference points as to where children are along the way to meeting the goals of the curriculum (e.g., Daro et al., 2011). They can provide a structure within which teachers can identify and address difficulties that arise for children. HLTs are seen as particularly useful for teaching concepts whose learning is problematic generally or for particular students (Simon & Tsur, 2004). One identifiable gap in the literature is the use of these trajectories for identifying and addressing the needs of high-achieving learners.

An example of the use of learning trajectories to develop teachers' work in assessing young children aged 5–8 years is provided by the Victorian Early Numeracy Research Project (ENRP). The three year project focused on developing teachers' understanding of mathematics in the early years, evaluating the effect of professional development programmes, and describing effective practice in mathematics in the early years of schooling (Clarke, 2001; Bobis et al., 2005). Central to the ENRP was the development of a framework of 'growth points' in young children's understandings of mathematics in different domains. Growth points were considered by the ENRP team as 'key stepping stones' along paths to mathematical understanding (Clarke, 2001). It was not considered that children would necessarily pass through each growth point in succession or that the growth points were discrete. Furthermore, the framework gave teachers a tool for assessing children's understandings and building on children's current skills and concepts. One of its purposes was to provide a basis for task construction for assessment via interview. In developing this framework the researchers drew on the work on learning trajectories. Assessment tasks were created to match the framework.

The issue of learning trajectories and assessment is also discussed in Chapter 6 in the context of assessing and planning for progression (Section: *Supporting Children's Progression with Formative Assessment*).

Supporting Learning for Pre-Service Teachers

We also see an important role for well-structured developmental progressions of concepts in the education of teachers, particularly at pre-service level. Detailed knowledge of these can provide pre-service teachers with frameworks related to general mathematics development.

Conclusion

In considering the potential of the range of work on learning trajectories, we find it useful to consider Fosnot and Dolk's (2001) perspective on the issues which are implicated in teachers' understandings of children's learning and of how to plan for that learning. Fosnot and Dolk argue that

> *Strategies, big ideas and models are all involved – they all need to be developed as they affect one another. They are the steps, the shifts and the mental maps in the journey. They are the components in a "landscape of the learning".* (p. 12)

The research indicates that teachers' understanding of developmental progressions is one aspect in helping them to develop hypothetical learning paths for use in their classrooms. They sit alongside their knowledge of the big ideas or key goals (see *Chapter 4*). They support teachers' understandings of children's emerging models (see this chapter). Research also suggests that teachers need a great deal of support in moving from a linear model of learning to one in which children engage as members of dialogic communities in tasks that are truly problematic (see Fosnot & Dolk, 2001). All of this has implications for teacher education, an issue that is discussed in Report No. 18 (*Chapter 6*).

The key messages arising from this chapter are as follows:

- Learning trajectories describe learning paths in the various domains of mathematics. These are based on developmental progressions which have been constructed for a number of big ideas in mathematics. They indicate a general sequence that might apply to development.
- There are different approaches to the explication of learning paths. For example, linear/nonlinear presentation, level of detail specified, mapping of paths to age/grade, and role of teaching. Different presentations reflect different theoretical perspectives.
- An approach to the specification of learning paths that is consistent with sociocultural perspectives is one which recognises the paths as
    - i. provisional, as many children develop concepts along different paths and there can never be certainty about the exact learning paths that individual children will follow as they develop concepts.
    - ii. not linked to age, since this suggests a normative view of mathematics learning.
    - iii. emerging from engagement in mathematical-rich activity.

Curriculum design must take into account the children's reasoning in and contribution to the learning-teaching situation.

CHAPTER 6

Assessing and Planning for Progression

This chapter looks at the assessment of mathematics and ways in which assessment data can be used in planning for progression in mathematics learning in preschool and primary school settings attended by young children. First, the chapter examines formative assessment in terms of conceptual underpinnings and key methods. The focus then shifts to diagnostic and summative assessment as the use of screening/diagnostic and standardised tests is considered. The chapter concludes with a consideration of the use of assessment data for planning and progression in a range of contexts, including immersion settings, and settings involving children with special needs.

The formative assessment methods discussed include observation, tasks, interviews, conversations and pedagogical documentation. The methods are inclusive of all children. Each of these provides scope for examining the embedded nature of children's mathematical learning, changes in their understandings, what children can do when supported by others, their potential capabilities and strengths, and their participation in activities and tasks. They also provide scope for assessing children's dispositions and identities as mathematics learners. These aspects are key foci of assessment from a sociocultural perspective (e.g., Fleer & Richardson, 2009). The discussion on screening/diagnostic tests urges care in using such methods, and highlights a need to draw on multiple sources of information when assessing children in various at-risk groups. Caution is advised in relation to the use of standardised tests with children in the 3–8 years age range.

Assessing Mathematics Learning in Early Childhood

Aistear (NCCA, 2009b) defines assessment as

> *the on-going process of collecting, documenting, reflecting on and using information to develop rich portraits of children and learners in order to support and enhance their future learning.* (p. 72)

Assessment in the Primary School Curriculum: Guidelines for Schools (NCCA, 2007) offers a similar vision of assessment.

In order to support children's learning, it is essential that teachers are familiar with each child's mathematical understandings and learning. Educators acquire this understanding through formative assessment of children's mathematical learning since this approach serves to best represent the complexity and depth of children's learning (e.g., Carr 2001; Carr & Lee, 2012; Drummond 2012; Perry, Dockett & Harley, 2007). Increasingly, assessment is seen as a collaborative process between children and adults, and one in which teachers support and scaffold children's work. This view of assessment is predicated on a view of pedagogy that has relationships at its core (e.g., Fiore, 2012).

Formative Assessment

Conceptual Frameworks

Eliciting children's mathematical thinking is critical to understanding, monitoring and guiding their mathematical learning. Research-based conceptual frameworks which describe mathematical thinking in terms of levels of sophistication (i.e., learning paths as described in *Chapter 4*), provide the basis against which educators can then interpret children's reasoning. This process of locating a child's thinking on what Battista (2004, p. 202) refers to as 'a detailed map of the cognitive terrain required to construct understanding of a topic' is referred to as cognitive-based assessment, and it is increasingly seen as an effective tool for planning learning opportunities and for guiding children in their construction of mathematical meaning. We also know that dispositional learning is a crucial aspect of early learning and this too must be monitored and fostered. Carr and Lee (2012) illustrate the centrality of dispositions when they state that 'Dispositions act as an affective and cultural filter for the development of increasingly complex knowledge and skills' (p. 15). In other words, children's dispositions towards mathematics and towards engaging in mathematical ways of thinking and knowing are influenced by how they feel towards these activities. Knowledge and dispositions develop hand in hand – they are interdependent. Formative assessment also plays a key role in the construction of a learner identity (e.g., Bruner, 1996; Carr & Lee, 2012). Identity develops as children interact with mathematical knowledge, skills and ideas in the home and in education settings – it is socioculturally constructed. This implies that children's identities as mathematics learners are formed during early childhood. Learning and a sense of identity cannot be separated; some consider them one and the same thing (Lave & Wenger, 1991). How teachers and parents recognise and respond to children's numeracy practices shape children's identities (e.g., Anderson & Gold, 2006). Educators can greatly influence the development of children's identities as mathematicians by the way in which they frame children's activity. For instance, children will bring a rich store of mathematical achievement with them to school. This needs to be recognised and harnessed. Carr and Lee (2012) remind us of the opportunities that educational settings provide not just for the construction of identity but also critically for the editing of learner identities. In other words, teachers can influence, in a positive way, children's perceptions of themselves as mathematics learners. One way to do this is for teachers to collect and study mathematics-related vignettes of children's social activities at home and in the education setting – and then to reflect on the meanings of these. This could be especially effective for supporting children during transitions and during the early months in a new setting, particularly when discussed with children and parents.

In the section that follows we focus on what research tells us about how caregivers and teachers can most effectively carry out assessments of learning in order to gather data on children's achievement and their developing dispositions and identities as mathematicians.

Methods

In line with good early childhood practice internationally, both *Aistear* (NCCA, 2009b) and *Assessment in the Primary School Curriculum: Guidelines for Schools* (NCCA, 2007) identify a range of appropriate methods of formative assessment including observations, conversations, tasks, tests and self-assessment. Educators can assemble portfolios of children's learning and they can work with children and parents to compile pedagogical documentation as evidence of children's mathematics learning. Effective assessment is closely related to teachers' knowledge and their recognition of what constitutes significant learning, some of which could be informed by their knowledge of general learning paths in the major mathematical domains. A number of methods can be used, often together, to build a rich picture of children's mathematical learning over time. The ability to recognise the mathematics in children's everyday activities and to extend the potential learning arising from these is critical.

Observations

Observations can provide educators with the data to write rich narrative assessments of children's mathematical learning. These assessments can focus on different aspects of children's mathematical development. Contextual information can be included in the emerging picture of children's development. Depending on the circumstances, questioning or follow-up tasks can be used in order to check children's levels of mathematical understanding demonstrated or assumed. In engaging in these processes, educators draw on their deep knowledge of what mathematics is and how it develops in early childhood (e.g., Ginsburg & Ertle, 2008).

Arising from observations, 'learning stories' (Carr, 2001) can be constructed by the educator or co-constructed by the educator and child/children, with contributions from family and other significant adults. These are narrative accounts of learning and development and they take a holistic approach to assessment. They are often supplemented with photographs.

Carr (2000) describes learning stories as 'structured observations, often quite short, that take a 'narrative' or 'story approach' (p. 32). They keep the assessment anchored in the situation or action. Learning stories are rich and deep accounts of selected events as they are observed through specific lenses, for example the themes or goals of the curriculum. These assessments are learner-centred as opposed to content-centred. They do not fragment children's learning and they pay attention to the positive, rather than focusing on need and deficit (e.g., Dunphy, 2008).

When initially developed, learning stories focused mainly on dispositional learning (e.g., Carr, 2001). However, recent developments of the method in early childhood classrooms in schools in New

Zealand have seen teachers focus on both knowledge and disposition. This involves the teacher noting both the mathematics and the learning disposition evident in the analysis (Carr & Lee, 2012). From their experiences in working with preschool- and school-based educators, these authors conclude that

> *Learning Stories can capture the intermingling of expertise and disposition, the connections with the local environment that provide cues for further planning, the positioning of the assessment inside a learning journey, and the interdependence of the social, cognitive and affective dimensions of learning experiences. At the same time, Learning Stories enable children and students to develop capacities for self-assessment and for reflecting on their learning.* (p. 131)

In addition, Carr and Lee argue that learning stories meet four challenges associated with formative assessment: the challenge of engaging children in co-authoring the curriculum and assessment and exercising agency in relation to aspects of their learning; encouraging reciprocal relationships with families; recognising learning journeys and continuities in learning over time; and appropriating a repertoire of practices where the learning is distributed over a number of languages and other modes of meaning making. Even the youngest children are now becoming everyday users of technology in the home and in early education settings (e.g., Plowman, Stephen & McPake, 2010). Learning mathematics with technology, and using technology to express mathematics understanding and thinking are increasingly important avenues of learning and expression for young children (see Report No. 18, Chapter 2, Section: *Digital Tools*). Arising from their work with teachers, Carr and Lee (2012) observe that

> *Learning Stories have now participated in the new digital technologies in three ways: transforming the ways in which Learning Stories can be constructed, tracing children's Information Communications Technology (ICT) learning journeys, and emphasising the value of image-based ways of thinking.* (pp. 112–113)

From the assessment perspective, this expands the ways in which children's learning can be identified and documented. It provides a multi-modal approach to assessment of children's mathematics learning.

Tasks

Tasks can be conceptualised in different ways; for instance, MacDonald (2011; 2012) draws attention to the value of mathematical drawing activities and of photographic assignments as tasks for assessing and extending children's understandings at the start of school. In schools, tasks are often initiated by the teacher and this in itself may present a challenge in ensuring that they are meaningful and relevant, and at the very least, motivating and engaging for young children. Educators need to consider the structure and characteristics of tasks and how these relate to the learning (e.g., Yelland & Kilderry, 2010). Tasks can be teacher-designed or they may be pre-designed ones that accompany curriculum materials. The key issue is that the teacher can identify the possibilities in the child's responses. Guidelines in relation to the use/development of tasks are presented in Report No. 18 (Chapter 2, Section: *Cognitively Challenging Tasks*).

Interviews

Interviews, or focused conversations, are opportunities to explore indepth children's thinking and reasoning through conversation (and observation), generally about tasks that the child undertakes as part of the interview. Observations, tasks and conversations during the course of an interview are methods that complement each other and they are frequently used together (e.g., Ginsburg, 1997b; NRC, 2009). The success of each is contingent on the teacher's knowledge and understanding of early childhood mathematics development (e.g., NRC, 2009). Some curricula in the United States, for example *Big Math for Little Kids* (e.g., Clements & Sarama, 2009b) and *Building Blocks* (e.g., Ginsburg, 2009b), have provided protocols for this work.

Ginsburg advises teacher interviewers to 'adopt, at least provisionally, a theoretical framework with which to interpret your observations' (1997, p. 120). Recently, he discussed how cognitive science can provide that framework in the shape of developmental trajectories or learning paths (Ginsburg, 2009b). He argues that understanding these provides a useful background to understanding individual children. But he also draws attention to the paradox of using developmental trajectories in interviewing:

> *The interviewer's goal is sensitivity to the child. The interviewer wants to have an 'open mind' in order to discover what is in the child's mind. The goal is to learn how the child thinks and how the child constructs a personal world…On the other hand, to discover something about the child's cognitive construction, the interviewer must have some ideas what to look for, some notions about the forms children's thinking may take. Lacking concepts for interpreting the child's behaviour and explanations, the interviewer is likely to overlook what is important and to focus on what is trivial.* (pp. 119–120)

As the educator engages with the child, assessments can be made: of performance, of thinking/knowledge, of learning potential, and of affect/motivation. The information derived can then be used to shape instruction 'in a principled way' (Ginsburg, 2009b, p. 111). The interview, well done, can detect strengths and weaknesses that otherwise may go undetected, but the ability to do the work well is predicated on well-developed mathematical as well as pedagogical subject knowledge. Mathematical knowledge for teaching is discussed in Report No. 18 (Chapter 6, Section: *Mathematical Knowledge for Teaching (MKT)*).

The understanding of the child's perspective, which is elicited in the course of the interview, provides a critical counter-balance to age/stage/level-related presentations of children's mathematical thinking and acknowledges the child as capable, knowledgeable, logical, sense-making and agentive. It recognises children as competent participants in their education (e.g., Dunphy, 2012). The interview is an opportunity for educator and child to co-construct mathematical understandings. Other significant gains are identified. For example, experience of the interview 'engages the child in talking about one's thinking, justifying one's conclusions, and in general engaging in mathematical communication' (NRC, 2009, p. 264). Ginsburg (1997b) too points to metacognitive and expressive gains: 'the child sharpens, or even acquires the ability to introspect and express thinking' (p. 114). These claims relate to the learning that can happen in the course of an assessment, what Wiliam (2007, p. 1054) refers to as assessment as learning. Because of its sensitivity to the individual, interviewing is particularly useful in seeking to accommodate a diverse range of mathematical abilities.

Conversations

While educators might quickly grasp the benefits of one-to-one interviewing, research has identified a particular need to provide educators with extensive curriculum guidance in interviewing for the purposes of promoting children's mathematics learning (NRC, 2009) often in the context of professional development (Ginsburg, 1997b). In reality, given the busy nature of classroom life, many educators may plan to use an extended interview on only a few occasions in any given year. Focused conversations may be the method of assessment used much more frequently. This method of assessment assumes knowledge of learning paths in different mathematical domains. While educators need to learn to use the interview as a means of making in-depth assessments of a child's understanding of a particular concept or big idea such as counting, more usually teachers also need to have mathematical conversations with children during the course of classroom activities as the opportunity occurs. For example, the child's understanding of shape can be ascertained in the course of activities with blocks or tangrams. Sensitive questioning and the use of a variety of questioning techniques is an area of general pedagogical knowledge that has been highlighted as a key factor in promoting early learning generally (e.g., Siraj-Blatchford et al., 2002). Donaldson's (1984) work illustrated the dramatic effect of the inclusion or omission of a single adjective in questioning children on so-called 'logical' tasks. Furthermore, it is essential that in questioning the youngest children we note her caution that *'the young child…first makes sense of situations (and perhaps especially those involving human intentions) and then uses this kind of understanding to help him make sense of what is said to him'* (p. 59). We know that questioning isn't the only way, nor necessarily the best way, of eliciting responses from young children (e.g., Fisher 1990; Norman 1993). The *Aistear* guidelines (NCCA, 2009b) identify a range of methods which the educator can use in interactions with young children. These include naming and affirming children's actions and behaviours; supporting participation and learning, and assisting learning. Interactions such as these present contexts for assessing early mathematics learning.

Pedagogical Documentation

Pedagogical documentation, the documentation of children's learning, is a framework for assessment which originated in the work of the Reggio Emilia preschools in Italy. Learning moments are captured usually through observation, transcription and visual/audio representations such as photos and recordings. This is the content of the pedagogical documentation. What makes pedagogical documentation different to traditional observation is the process that takes place in the collaborative negotiation and revisiting of the learning. Pedagogical documentation may be defined as:

> *both content and process involving the use of concrete artefacts in the form of audio recordings, photographs, examples of the children's work, and collaborative revisitation, interpretation, and negotiation by the protagonists (children, teachers and parents) to promote dialogue and reflection.* (MacDonald 2007, p. 233)

While the approach seems theoretically to have great potential, there are few if any published examples of its use in the area of mathematics learning and teaching. In the study reported here, it proved challenging for teachers working with early literacy in Canadian schools due to the need for high levels of teacher support. However, the examples of the documentation process offered from the Reggio perspective do include some mathematically focused work, for example Shoe and Meter. On the Reggio Children website the project is described thus:

> *The starting point is a concrete request: the school needs a new table. Teachers propose to children to take care of it: what to do? The first approaches to the discovery, to the function and the use of measures. Children have access to the mathematical thinking through the operations of orientation, play, choice of relational and descriptive languages. (http://www.reggiochildren.it/?libro=scarpa-e-metro&lang=en)*[9]

Supporting Children's Progression with Formative Assessment

In rural and regional Australia, research aimed at investigating early childhood educators' thoughts on young children's mathematical thinking and development found that, while preschool teachers were learning and keeping records in relation to mathematics, it didn't extend beyond observation. Participants in that study also reported reluctance to introduce technology into the settings and this was due to their lack of confidence and competence (Hunting et al., 2013). This is significant given the importance that Carr and Lee (2012) accord to technology in identifying and documenting early learning (see discussion of methods above). It is quite likely that similar attitudes are to be found amongst the educator population here. In the United States, the NRC report (2009) notes that while

9 Reflecting on the work of Castagnetti, M. & Vecchi, V. (1997) *Shoe and meter.* Reggio Emilia, Italy: Reggio children.

formative assessment shows great promise, the methods of assessment have not been clearly linked to the teaching that takes place subsequently. A number of mathematics educators suggest that some of the challenges of integrating learning, teaching and assessment can be met by reference to learning and teaching paths. Ascertaining children's learning and their multi-path learning trajectories enables educators to make judgements regarding how best to support future learning. For instance, Ginsburg (2009b) argues that the rich information gained from one-to-one interviews, which may include insights into children's experiences with aspects of mathematics in everyday situations, actually reveals a great deal about children's understanding of mathematics and this information can be used to compile a profile of the child as a mathematics learner. The teacher can then design appropriate learning experiences for the child. As Ginsburg describes it, the teacher can do so since he/she is now in a position to decide 'on a specific course of action with a specific child' (p. 125). In other words, the teacher is now in a position to decide on a teaching path to help the child. Ginsburg sees the teacher's judgement at this point as critical and one that cannot be replaced by a pre-designed script. As he sees it '…the task of teaching mathematics is so complex that a detailed script is likely to do more harm than good' (p. 126).

From a RME perspective, mathematics educators have identified intermediate steps for trajectories in the areas of number, measure and geometry as guidance for assessment. They argue that these ensure that teachers know what to look for (van den Heuvel-Panhuizen, 2008; van den Heuvel-Panhuizen & Buys, 2008). Learning-teaching trajectories as a basis for assessment are discussed further in Report No. 18 (Chapter 3, Section: *Content Areas*). Earlier, we discussed the role of developmental progressions as a support for teachers in assessing learning (see *Chapter 5*). Young-Loveridge (2011) describes how, in New Zealand, individual diagnostic assessments (based on interviews), in conjunction with a research-based framework outlining the learning progression in number, have provided a powerful means for teachers to determine children's starting points and make decisions about ways to enhance learning.

A project which sought to improve mathematics and numeracy outcomes through a sustained, collaborative programme of professional development and action research was carried out in 2004 in South Australia. As part of that project, Perry, Dockett and Harley (2007) worked with preschool educators who engaged in writing learning stories which focused on children's 'powerful mathematical ideas' (see *Chapter 4*). They did so in the context of eight developmental learning outcomes for children's learning in the preschool year as presented in *The South Australian Curriculum, Standards and Accountability Framework* (Government of South Australia, 2001). The findings established the technique of learning stories as a valid assessment method compatible with the holistic approach inherent at the preschool level. The researchers describe how this was achieved through the educators' use of a numeracy matrix. The matrix constructed by the researchers and the educators consisted of 56 cells (8 developmental learning outcomes x 7 powerful mathematical ideas), with each cell of the matrix providing examples of pedagogical questions for the educators as they were teaching towards, assessing or reporting on the developmental learning outcomes. For instance, in relation to the powerful mathematical idea of *Algebraic Reasoning* and the developmental learning outcome that

Children develop a range of thinking skills, the questions generated were 'How do we encourage children to use patterns to generate mathematical ideas?' and 'In what way do we provide opportunities for children to reflect upon their mathematical pattern making?'.

The matrix proved to be a powerful tool for enabling mathematically-focused assessment practices. It appeared that the educators used the matrix as a framework for reflecting on and identifying children's mathematical learning. They used it as a scaffold with which to build the analyses of children's activities, and subsequently to write the resulting learning stories. Significantly, the stories captured both dispositional and content-related learning, and documented learning in relation to both. As observed by the researchers:

> *the matrix is a dynamic reflection of the knowledge of the educators using it, and, as such, should be expected not only to be grounded in the contexts in which these educators work but to change as their knowledge grows.* (Perry, Dockett & Harley, p. 5)

The methods reviewed above will undoubtedly prove challenging for teachers. Nevertheless, if there is to be coherence between mathematics curriculum, pedagogy and assessment, it is clear that educators will need to be supported and encouraged to move towards implementing such approaches in assessing early mathematics learning.

Arising from the above discussion, there are three important themes evident in relation to assessment of early mathematics. They are as follows:

- The role of strong conceptual frameworks such as general developmental progressions when assessing. These determine what teachers recognise as significant learning, what they take note of and what aspects of children's activity they give feedback on.
- The possible benefits of co-constructing assessment with children.
- The potential of digital technologies for documenting learning and for shaping learner identities.

Diagnostic and Summative Assessment

There is considerable debate in the literature on the value of administering more formal measures of early mathematical knowledge, whether those measures comprise screening/diagnostic tests designed for small groups or individuals that are administered using standardised procedures, but mainly produce qualitative information, or more formal standardised tests which are administered to larger groups and almost always lead to norm-referenced interpretations. Indeed, standardised testing in particular has generally been rejected by early childhood educators as a valid assessment approach for use with young children. This position is encapsulated in the following statement by Fiore (2012):

> *In current early childhood classrooms, most assessment is designed to acquire information that will help responsible individuals make decisions in the interest of the child's growth and development. Testing as part of such assessment takes time and resources…This mandatory time either reduces classroom time for free play and exploration or must be carved out of other organized periods of the day…The assessment process is further challenging because teachers recognize that one particular test or score does not paint a full, clear picture of a complex, developing child. This is supported by research that states that standardized testing of children under the age of 8 is scientifically invalid and contributes to detrimental labelling and can permanently damage a child's educational future…* (p. 5)

A key consideration in relation to such tests concerns the aspects of early mathematics measured (that is, what, according to the test, constitutes mathematical knowledge). According to Smith-Chant (2010b), early numeracy tests often measure skills found on the mathematics curricula taught in the early primary years, and may afford limited attention to important preschool numeracy skills that may be foundational for later mathematical development. Such tests may overestimate the formal aspects of numeracy knowledge, particularly in the areas of number-language and arithmetic, and under-estimate the non-language-based aspects of numeracy understanding (e.g., the concept of non-verbal counting, more, less, time and patterning). Moreover, they may have a heavy language component, presupposing that a child's understanding of early numeracy is language-based.

Snow and Van Hemel (2008) outline some key issues than can arise in administering direct assessments such as diagnostic tests and more formal standardised tests. These include the following:

- The child may not be familiar with this type of task or be able to stay focused.
- Young children have a limited response repertoire, being more likely to show rather than tell what they know.
- Young children may have difficulty responding to situation cues and verbal directions.
- Young children may not understand how to weigh alternative choices, for example, what it means for one answer to be the 'best' answer.
- Young children may be confused by the language demands, such as negatives and subordinate clauses.
- Young children do not respond consistently when asked to do something for an adult.
- In some cultures, direct questioning is considered rude.
- The direct, decontextualised questioning about disconnected events may be inconsistent with the types of questions children encounter in the classroom.
- Measurement error may not be randomly distributed across programmes if some classrooms typically use more direct questioning, like that found in a standardised testing situation.

Berliner (2011) argued that many young children may have a restricted ability to comprehend the formal, spoken instructions required for many standardised tests, that they lack the sophistication to interpret situational cues or written instructions, and that a test administered at one point in time may not capture important shifts in changes in a child's development.

While it is accepted that diagnostic and summative assessments may not be appropriate or desirable for use with young children, we recognise that there are contexts in which their use may be seen as helpful (for example, to identify children who may be at risk of learning difficulties). The key issue here is that, if used at all, they should be used as only one measure of children's mathematics learning and development. Next, we consider the types of information that screening/diagnostic tests and standardised tests can provide.

Screening/Diagnostic Tools

The primary purpose of screening/diagnostic tests is to identify children's learning difficulties in mathematics at an early stage, with a view to providing early intervention. Such tests are often administered on a one-to-one basis, allowing test administrators (usually teachers) to evaluate children's responses to set tasks, including the reasoning behind those responses. In relation to the lowest-achieving children, four components of number competence have been highlighted as important to include in screening/diagnostic measures. These are (i) magnitude comparison, or the ability to discern which number in a set is greatest, and relative differences in magnitude; (ii) strategic counting, defined as the ability to understand how to count efficiently and use counting strategies; (iii) ability to solve simple word problems; and (iv) retrieval of basic arithmetical facts (Gersten et al., 2012).

The following are issues that may arise in the administration and interpretation of screening/diagnostics tests, such as *the Drumcondra Tests of Early Numeracy* (ERC, 2011) and the *Learning Framework in Number* (LFIN) (Wright, Martland, & Stafford, 2006):

- Such tests are generally administered to children deemed to be at-risk of learning difficulties in mathematics; hence, not all children in a group will need to be assessed using these methods; indeed many screening/diagnostic tests are not designed to provide detailed information on the abilities of average or higher-achieving learners.
- Screening/diagnostic tests can provide valuable qualitative (formative) information on the reasons underlying children's responses, if test administration allows users to gather and record such information.
- Such tests are often linked to instructional programmes or interventions. In some cases, the interventions have been demonstrated to be effective in a range of contexts; in other cases, there may be limited evidence to support instructional recommendations, and hence care will need to be exercised in deciding what support to provide.

- Performance on screening/diagnostic tests (and on other types of tests) may be associated with factors such as educational disadvantage or the children's linguistic skills, and these factors need to be taken into account in interpreting outcomes.
- Performance on screening/diagnostic tests can be predictive of later performance on more formal standardised measure of mathematics (e.g., ERC, 2011). However, such tests may not be predictive at the individual child level, and other evidence, in addition to the outcomes on a screening/ diagnostic test, may need be taken into account in making inferences about a child's risk status.
- Screening/diagnostic tests for young children often focus on number, and other important aspects of numeracy or mathematics, such as shape and space, may be overlooked.

Standardised Norm-Referenced Tests

In general, group-administered, standardised tests of numeracy or mathematics are deemed inappropriate for use with young children. Indeed, in the US, states are not required to administer standardised tests for accountability purposes until children are in the latter part of third grade (8–9 years of age). Similarly, while assessment at Key Stage 1 in England originally comprised formal paper-and-pencil tests in mathematics, this is no longer the case, and teachers now submit results based on their own professional judgements, though supports are available to help teachers make judgements, including optional tests.

In Ireland, standardised tests are administered to children in second class, as part of the National Assessments of Mathematics Achievement (see Eivers et al., 2010), which is conducted every five years. In addition, since 2012, schools are expected to administer standardised tests to children in second, fourth and sixth classes, and to report the outcomes to parents and to the school's Board of Management. Schools may exempt certain children from testing and/or reporting, though criteria for this are not well defined. Drummond's (2012) analysis of the test performance of a young boy named Jason (aged 7 years 6 months), provides a graphic account of the inadequacies in using such tests with children of this age as a way of assessing individual learning.

While standardised tests can provide an overall indication of a child's performance (for example, a standard score, percentile rank or sten score), and some of these scores can be aggregated across children at the same class level (e.g., the proportions of children in a class scoring at each sten score), they provide limited diagnostic information, and, where such information can be generated, it may be distorted because the tests have to serve multiple purposes.

Although standardised tests are typically based on a framework that broadly mirrors the underlying curriculum, there may be limited value in relying on content-area or process subtest scores. This is because, in general, there may be too few items on a subtest to allow for reliable information to be generated. This often tends to be the case with the Data strand, which may be represented by just a few items on a test for young children.

The recent increased emphasis on standardised testing for accountability purposes (e.g., DES, 2011) may also lead to an increased emphasis on preparing children to take standardised tests (meaning that test content becomes very familiar to children over time). A consequence of this is that test performance may improve, but children's proficiency in mathematics may not change.

Finally, standardised tests do not provide information on such factors as procedural fluency (accuracy, efficiency and flexibility), strategic competence, adaptive reasoning (logical thinking and justification) or productive disposition (behavioural-emotional components) (Mueller, 2011). In other words, standardised tests tell us nothing about these key strands of mathematical proficiency. Clearly, where used, standardised tests can only be considered to comprise one element of a more comprehensive assessment framework for planning, teaching, and learning of mathematics and which has at its centre a strong practice of formative assessment.

Many of the issues that arise in administering and interpreting the outcomes of group-administered standardised tests also apply to individually-administered standardised tests. These include the range of mathematical knowledge assessed and the lack of information on children's thinking processes. However, an individually-administered test does allow for the creation of an easier rapport between test administrators and child, than is possible with a group-administered test.

Planning for Progression Using Assessment Outcomes

A primary purpose of gathering assessment information is to use it as a basis for planning instruction. Where the mathematical development of young children including preschoolers is concerned, adults will need to draw on the outcomes of appropriate forms of formative assessments – observations, tasks, interviews and conversation. The interpretation of outcomes is guided by the adult's understanding of children's general cognitive development (what should be expected at different developmental points in terms of language and understanding), as well as mathematical development (e.g., through familiarity with learning paths – see above). As outlined in *Aistear* (NCCA, 2009b), a key aspect of assessment is the recording of assessment data so that adults have a basis on which to plan future learning activities, taking into account children's current knowledge and their needs.

Planning for progression will occur at the level of the individual teacher/carer, and among groups of adults working with or at least familiar with the same children. The literature (e.g., Ginsburg, 2009a) suggests that, for younger children, the focus is on

- the mathematical knowledge that children bring from home (including invented strategies) and how this relates to opportunities for mathematical development presented in preschool/early primary school
- the quality of their everyday language and their mathematical language, including their knowledge and use of key terms in areas such as number and shape and space

- their ability to talk about problem-solving in formal and informal mathematical activities
- their understandings and metacognition with respect to mathematics – their sense of their own ability to solve a mathematical problem
- their abilities to make connections across aspects of mathematics, and between mathematics and everyday life
- their mathematical dispositions.

As children progress through the primary school classes, teachers may extend the range of assessment outcomes that they use in their planning to include those arising from screening/diagnostic tests, and, perhaps towards the end of the 3–8 years range, from standardised tests. At this stage, it is important to integrate the outcomes of formative and diagnostic/summative assessments since, as Ginsburg (2009b) points out, standardised tests, in and of themselves, do not provide information about children's underlying thinking processes. In this view, children might do quite well on a standardised measure, yet may lack the sense-making and critical thinking that are the hallmarks of mathematical proficiency.

Immersion Settings

One immersion setting in the Irish context is the Irish-medium setting – whether naíonraí[10] or primary schools – where children may learn mathematics in a first, second, or third language – Irish. Here, teachers will have to take children's proficiency in Irish into account in interpreting assessment outcomes – does the child have sufficient language proficiency to understand the task being assigned, and to express his/her mathematical thinking (see also Chapter 3, Section: *Variation in Language Skills and Impact on Mathematics*). There is, for example, evidence from the 2010 national assessments of English and mathematics in Irish-medium schools (Gilleece et al., 2012) that children may have struggled with the language on a standardised test of mathematics administered in Irish, and hence may have performed less well than they were capable of. This, perhaps, underlines the importance of combining information from multiple sources in arriving at inferences about the mathematical performance of children in such settings.

Wood and Coltman (1998) argue that 'it is difficult to over-emphasise the importance of verbal communication in the development of children's mathematical understanding' (p. 114). The implication of this is that empowering children to develop their language skills in the language of instruction of the school is of vital importance for supporting the development and expression of mathematical understanding. Many children in Irish-medium settings have a common language, English, shared among themselves and the teacher. This facilitates communication, even though the

10 Naíonra is a playgroup run through Irish for children aged 3–5 years, who attend daily for 2–3 hours.

language policy of the settings may be discouraging of this. Code-mixing, where utterances involving vocabulary or structures from two or more languages are combined, is often used as a strategy to allow communication and understanding (Mhic Mhathúna, 1999).

Similar issues arise in addressing the assessment needs of children whose first language is not English or Irish. Commentators differ on the need to provide assessments for the child in their stronger language (Baker, 2001), or in the language of instruction (Sierra, 2008). Peal and Lambert (1962) established that proficiency in both languages resulted in higher scores in verbal and non-verbal testing of intelligence, an early forerunner to Cummins' threshold theory (Cummins, 1976; Cummins 2000). The threshold theory suggests that bilingual children who have achieved a level of competence in both languages are afforded a cognitive advantage in all other areas of the curriculum. Conversely, children who have not reached a minimum standard of competence in both languages may experience negative cognitive and academic outcomes, with obvious implications for mathematics learning. While the threshold theory has been criticised for failing to clarify in concrete terms what these thresholds are (Chin & Wigglesworth, 2007), or for equating academic success with cognitive ability, without allowing for factors such as socio-economic status (MacSwan, 2000), evidence to support it has emerged from the US (Kessler & Quinn, 1982), Ireland (Ní Ríordáin & O'Donoghue, 2007), Malta (Farrell, 2011), and Papua New Guinea (Clarkson, 1992).

This suggests that age-appropriate levels of language competence in both languages should be considered when forming assessment opinions of children's achievements in mathematics. Educators carrying out assessment procedures such as interviews, observations or tasks in an immersion context have the dual purpose of assessing and evaluating both the mathematical competences and language competences of the child to gain a full picture. Dual language assessment (Murphy & Travers, 2012) is particularly important in this context, though it should be recognised that this adds to the complexities of the process, and to the demands on the child. When developing assessment materials or guidelines in a dual language context, care needs to be taken to ensure that tasks or questions on both language forms are developed collaboratively by translation and education experts to ensure their validity in both languages and minimise the danger of dealing with unfamiliar vocabulary or language constructions in either language, which would hinder the expression of mathematical knowledge or thinking on the part of the child (Rogers, Lin & Rinaldi, 2011). It might also be considered that difficulties that immersion children experience in mathematics may best be addressed not only by interventions aimed at supporting mathematical concept and skill acquisition, but also by interventions aimed at raising general language competence in both languages.

Children with Special Needs

The assessment of the mathematical and other abilities of children with special educational needs is complex. According to Snow and Van Hemel (2008):

- It is important to use multiple sources of information in arriving at decisions about the needs of children with special education needs, as the performance and behaviour of children with special needs across settings and situations can be even more variable than those of typically developing children.
- The variability in the performance of children with special needs across situations requires incorporating information from family members to obtain an accurate picture of the child's capabilities.
- A key principle applicable to all children but of special relevance to children with special needs is the importance of providing them with multiple opportunities to demonstrate their competencies.
- The setting for the assessment, the child's relationship with the person conducting the assessment, the ability of the assessor to establish rapport, fatigue, hunger, interest level in the materials and numerous other factors could result in a significant underestimation of the child's capabilities.
- Many young children with special needs are not capable of complying with all of the demands of testing situations, arising from lack of language, poor motor skills, poor social skills, and lack of attention and other self-control behaviours.
- Assessment tools should have a low enough floor to capture the functioning of children who are at a level far below their age peers.
- In assessing young children with special needs, it is important to consider the test's assumptions about how learning and development occur in young children and whether these are congruent with how development occurs in the child being assessed.

Therefore, considerable care needs to be exercised in the use of formal approaches to assessment with young children with special needs. In such circumstances, appropriate formative assessment methods may present the best solution. In relation to this, Douglas et al. (2012) note that, on occasion, child-led assessment through conversation methods may be problematic. It has also been suggested that, for some children with special needs, attentiveness should also be a focus of assessment (e.g., Gersten et al., 2012), as these children may not have the attention required to concentrate on the task in hand.

Conclusion

In relation to assessing and planning for progression in children's mathematical development, a number of approaches were reviewed, including the use of formative, diagnostic and summative assessments. Formative assessment methods are seen as coherent with the image of children as active powerful learners who learn mathematics as they engage in everyday activity with parents/caregivers, peers and teachers. Screening and diagnostic approaches are seen as useful for recognising and supporting children who are having difficulties with mathematics. Attention is drawn to the inappropriateness of standardised tests and their inability to adequately portray the mathematical learning and development of young children.

The key messages in this chapter are as follows:

- Of the assessment approaches available, formative assessments offer most promise for generating a rich picture of young children's mathematical learning.
- Strong conceptual frameworks, including a sound understanding of general developmental progressions (learning paths), are important for supporting teachers' formative assessments. These determine what teachers recognise as significant learning, what they take note of and what aspects of children's activity they give feedback on.
- There is a range of methods (observation, tasks, interviews, conversations, pedagogical documentation) that can be used by educators to assess and document children's mathematics learning and their growing identities as mathematicians. These methods are challenging to implement and require teachers to adopt particular, and for some, new, perspectives on mathematics, on mathematics learning and on assessment. Digital technologies offer particular potential in this regard.
- Constructing assessments which enlist children's agency (for example, selecting pieces for inclusion in a portfolio or choosing particular digital images to tell a learning story) has many benefits, not least of which are the inclusion of children's perspectives on their learning and their assessments of their own learning.
- More structured teacher-initiated approaches and the use of assessment within a diagnostic framework may be required on some occasions, for example, when children are at risk of mathematical difficulties.
- The complex variety of language backgrounds of a significant minority of young children presents a challenge in the learning, teaching and assessment of mathematics. Children for whom the language of the home is different to that of the school need particular support in developing language in order to maximise their opportunities for mathematical development and their participation in assessment.
- Educators carrying out assessment procedures such as interviews, observations or tasks in an immersion context have the task of assessing both the mathematical and language competencies of the child to gain a full picture of their development. Dual language assessment is particularly desirable in this context. This applies to both EAL and to Irish-medium settings.

CHAPTER 7

Addressing Diversity

In line with the inclusive nature of the perspective adopted in this report, it is important to reiterate the assumption that mathematics is relevant to all children and that each child has the right to access, participate in and benefit from enriching mathematical experiences. In discussing the literature and perspectives on children who experience difficulties learning mathematics, there can be a disproportionate emphasis on gaps and needs. However, we would preface this chapter by stressing that all children have strengths and preferences in relation to mathematics and that the goal is always to support the child through using these strengths and preferences. It is also important to understand what we do not know in relation to mathematical development and take the perspective that any perceived difficulties and delays are the responsibility of the teacher and school to address. The groups of individuals who often require particular attention in the teaching and learning of mathematics are 'exceptional' children (those with developmental disabilities or who are talented mathematically), children for whom English is not a native language or those living with disadvantage. In this chapter, an overview is given of the different ways exceptional children are grouped and how attention might be given to their particular needs. Some consideration is also given to addressing cultural diversity in mathematics learning. In essence, it is to be argued that mathematics teaching that is sensitive to and appreciative of individual and/or group variation is effective 'mathematics for all'.

Identification of Learning Difficulties in Mathematics

Butterworth (2005) claims that 'specific disorders of numeracy are neither widely recognised nor well understood' (p. 12). Attempts to categorise and label children experiencing low achievement/learning difficulties in mathematics have been problematic. Such approaches underestimate the role of instruction and experience in the development of critical knowledge and skills. It often assumes that, because children are in the same class with the same teacher, this can be controlled for. However, a myriad of influences affects how children construct knowledge and interact and engage with a teacher and their environment, or not. Also, it is very difficult to isolate the influence of inappropriate teaching or home and preschool experiences on low achievement in mathematics. Data on low achievement often do not distinguish between a delay, a temporary difficulty and more persistent long-term difficulties in the subject. Given this uncertainty, Dowker (2004) recommends that 'ultimately, the criteria for describing children as having 'mathematical difficulties' must involve not only test scores, but the children's educational and practical functioning in mathematics' (p. i).

A response to the above uncertainty with definitions and criteria in the US has been the development of the Response to Intervention (RTI) initiative. This arises out of the importance of monitoring the effectiveness of mathematical teaching and learning prior to classification of a learning disability. The primary goal of RTI is the prevention of difficulties through tailored evidence-based interventions. A secondary goal is the use of the data on progress with the intervention for the referral and identification of students with specific learning disabilities. It is now part of the US federal law in this area.

Exceptional Children

Kirk et al. (2012) define as 'exceptional' a child who differs from the 'typical' child in (i) mental characteristics, (ii) sensory abilities, (iii) communication abilities, (iv) behaviour and emotional development and/or (v) physical characteristics. The term includes both the child with developmental delays and the child with gifts and talents. In their view:

> *Individuals with exceptionalities help us better understand human development. Variation is a natural part of human development; by studying and teaching children who are remarkably different from the norm, we learn about the many ways in which children develop and learn. Through this knowledge, we inform ourselves more thoroughly about the developmental processes of all children.* (p. 3)

They remind us that the term 'typical' is problematic (that is, each of us differs from others in some regard) but, from an educational perspective, 'exceptional' usually suggests a learner for whom some modification has to be made to accommodate his or her individual needs. Notwithstanding this, there is broad consensus that 'distinctive teaching approaches' are not required for exceptional learners, although there is a need to address individual needs (e.g., Davis & Florian, 2004).

Intellectual and Developmental Difficulties

In reviewing the literature, distinctions are made between children with specific difficulties in mathematics, and children with difficulties with components or sub-components of mathematics. In addition, children can have difficulties arising from or as a risk factor because of a disability, specific or general.

Specific Difficulties in Mathematics

Difficulties in learning mathematics have been recognised for at least a century (Siegler, 2007). Multidisciplinary research of the issues has increased in recent decades but it lags substantially behind the equivalent level of attention afforded to literacy difficulties. Likewise, the evidence base is not as strong as for reading. However, there is consensus that a significant number of children exhibit poor achievement in mathematics (Swanson, 2007).

In studying the nature of difficulties in mathematics, Dowker (2004) emphasises the crucial understanding that arithmetic is not a single entity but is made up of many components. By arithmetic ability is meant

> *knowledge of arithmetical facts; ability to carry out arithmetical procedures; understanding and using arithmetical principles such as commutativity and associativity; estimation; knowledge of mathematical knowledge; applying arithmetic to the solution of word problems and practical problems; etc.* (p. ii)

Studies highlight that it is possible for children to show marked discrepancies between components of arithmetic. Dowker (2004) concludes that 'children, with and without mathematical difficulties can indeed have strengths and weaknesses in almost any area of arithmetic' (p. 5). Despite this variability, research has pinpointed some areas of mathematics that create more problems for children than others, though there is less agreement on the underlying mechanisms underpinning these patterns.

There is much written on the nature of the difficulties that children can have and comparisons with children without such difficulties. Siegler (2007) highlights a number of promising developments in the field regarding the structure of mathematical disabilities: Geary et al. (2007) highlight the role of three processes: working memory functioning, phonological processing and visuo-spatial thinking; Butterworth and Reigosa (2007), working from the perspective of neuro-imaging, which often shows different results to other approaches such as interviewing, suggest that domain-specific modular representations of number play a role, and that there is little evidence supporting the role of working memory. They suggest that children at risk of mathematical difficulties are slower at subitising (saying how many are in a small set without counting). An increasing number of researchers such as Jordan (2007), Barnes et al. (2007) and Bull (2007) emphasise the role of poor mastery of number facts and fact retrieval, poor number sense and weaknesses in conceptual understanding as underlying problems. In addition, researchers highlight social and emotional influences such as motivation and maths anxiety.

Developmental Delay

Berch and Mazzocco (2007) make a distinction between children with developmental delay and those with a mathematical learning disability. Dowker (2004), in a review of the area, highlights that

> *a significant number have relatively specific difficulties with mathematics. Such difficulties appear to be equally common in boys and girls, in contrast to language and literacy difficulties which are more common in boys.* (p. i)

This raises the question as to the differences between children with specific difficulties with mathematics and those with non-specific difficulties associated with low achievement in general. There has been inconsistency in the findings related to this question but more evidence is pointing to specific difficulties being milder than difficulties associated with low achievement in general.

Children with certain disabilities can experience difficulties in mathematics. Research has highlighted, for example, difficulties for children with specific language impairment (Donlan, 2007); Turner and fragile X syndromes (Mazzocco et al., 2007); spina bifida (Barnes et al., 2007); attention deficit hyperactivity disorder (Zentall, 2007) and with brain injuries (Zamarian et al., 2007). Mathematical difficulties often co-occur with dyslexia and language difficulties (Dowker, 2004). While children with some forms of brain damage or genetic disorder can have disproportionate difficulties with number, on the whole, children with general learning disabilities display similar developmental profiles as peers of the same mental age (Dowker, 2004).

Porter (1999) makes the distinction between what children can do and what they understand. In a study comparing the performance of children with severe learning disabilities and nursery-school (i.e., preschool) children, Porter (1998) found no difference in performance on simple counting and error-detection tasks. However, there was a difference in the acquisition of counting skills. Porter outlines four profiles of performance: non-counters, acquirers, transitional, and error detectors. Mental age proved to be the best predictor of performance tested. The pattern of attainments of the children described as acquirers differed from that of preschoolers in that adherence to the one-to-one principle was easier than adherence to the stable-order principle for both small and large sets. The performance of the children suggested that it was necessary to learn the skills of counting prior to understanding what it means to count.

In a review of the literature on deaf children and mathematics learning, Nunes (2004) concludes with a hypothesis that 'deafness is a risk factor for difficulties in learning mathematics rather than a cause' (p. 151). In addition, findings suggesting that lack of informal mathematical experience may have a 'wide-ranging effect on deaf children's logical and mathematical development' (p. 155) are presented. In experiments in problem-solving, Nunes found 'that there is a gap between hearing and deaf children's use of actions to solve problems and that this gap is often more severe when the actions have to be coordinated with counting' (p. 154). Marschark and Spencer (2009) conclude that:

> *Delays in language development, a relative lack of exposure (incidentally and in classrooms) to life-based problem-solving activities, and frequently inadequate pre-service teacher preparation in mathematics are believed to lead to the overall delay in development of maths concepts and skills by students with hearing loss. Below-age language skills limit access to teacher-provided as well as text-based explanations and most deaf and hard-of-hearing students lack age-appropriate command of technical vocabulary in mathematics.* (pp. 139–140)

Mathematically Talented Children

In TIMSS 2011 mathematics, in which fourth class children in over 50 countries participated, the percentage in Ireland reaching the Advanced International Benchmark, while twice the international median, was well below the percentages in the top three performing countries (Singapore, the Republic of Korea, and Hong Kong), and also well below the percentages for Northern Ireland and England (Eivers & Clerkin, 2012). This suggests that many children in Ireland may not reach their potential in mathematics, compared with their counterparts in high-scoring countries. There is broad consensus that, internationally, the needs of children who are advanced (talented) in mathematics are not met (Diezmann et al., 2004). For this reason, these children 'underachieve' and are at risk of becoming quietly disaffected from mathematics in future years (Nardi & Steward, 2003). Such children may, however, be quite advanced in different mathematical domains, e.g., in a capacity to reason analytically or spatially (or perhaps both), and teachers need to be sensitive to the varying needs of these children (Diezmann & Watters, 2002). In particular, the needs of these children can be met by the provision of challenging tasks that have scope for learning and the use of metacognitive skills (ibid). However, this is not a case for 'streaming' or for a differentiated curriculum. Rather tasks can be created that allow all children some form of success. In this regard, Sohmer et al. (2009) speak of tasks with 'high-level cognitive demands'. Such tasks are characterised by 'multiple entry points, solution strategies and interpretive claims' (p. 112) and allow different students to access them in a variety of ways. Fiore (2012) refers to these tasks as 'tiered assignments':

> *The idea behind tiered assignments is to provide students with parallel tasks that have different levels of depth, complexity, and abstractness, as well as different support elements or explicit guidance. All students work toward the same goal or outcome, and the differentiated tasks allow students to build on their prior knowledge and strengths while their work on the tasks provides them with appropriate challenges.* (p. 143)

In redeveloping the mathematics curriculum for 3- to 8-year-olds, consideration needs to be given to the design and development of such tasks.

Cultural Variation

Wright (1994) describes a 3-year difference in the numerical knowledge of children as they begin primary school. Some 4-year-olds have attained a knowledge of number that some of their peers will not attain until they are 7 years old. Griffin et al. (1994), using a standardised test of children's conceptual knowledge, also found a 3-year gap in performance among 5- to 6-year-olds, with children from low-income communities performing like middle-income 3- to 4-year-olds. Without any intervention, such gaps widen throughout primary school. The Cockcroft report (1982) found that in a class of 11-year-olds, there is generally likely to be a 7-year range in arithmetical ability.

The general view espoused in this report that a teaching approach that is linked to meaningful cultural referents and that assumes that all children have the capacity to engage successfully in mathematics is an effective approach for all children regardless of their gender, ethnicity or social class (e.g., Ladson-Billings, 1995). It is also assumed that individual variation is the norm and not the exception (Fiore, 2012).

Ethnicity

In the 2009 *National Assessments of Mathematics and English* (Eivers et al., 2010), one of the factors associated with lower child achievement (in both English and mathematics) was that of speaking a first language other than English or Irish. Indeed, children are often perceived to be experiencing difficulty in mathematics on the basis of their relatively poor performance in achievement tests. However, such comparisons – while perhaps useful in terms of highlighting disparities – focus on access and achievement from a dominant perspective (generally white, middle-class students) and thereby preserve the status quo (Gutiérrez & Dixon-Román, 2011). A particular problem is that equity issues tend to be discussed from the perspective of group differences. Secada (1995) puts it like this:

> *[T]he search for group differences grants legitimacy to the view that diverse student populations are somehow deficient, exotic, or primitive when measured against the dominant norm. However, if all one can write or speak about is how a specific group is different from the norm, then the results are an impoverished view of that group and the validation of the belief that equity groups are somewhat inferior.* (p. 153)

Fiore (2012) exhorts the need for teachers to be 'culturally responsive' and decries the effect of test scores on such practices:

> *In an ideal situation, culturally responsive teachers, curricula, and assessments would support children's diversity, but the reality of pressures to produce evidence of annual yearly progress means that children's learning styles, language, temperaments, and identities are viewed as potential obstacles to successful assessment scores and ratings.* (p. 128)

Dooley and Corcoran (2007) argue that, while distinctive teaching approaches for different groups can lead to a deficit view of mathematics learning, the notion of 'one curriculum for all' might also perpetuate inequalities. Malloy (1999) proposes that pedagogy, not content, must become multicultural. This means valuing the many ways that children make sense of mathematics. Some means by which this might be achieved by teachers include ensuring that connections between new and old ideas are evident to the learner, using problem contexts that are meaningful to the child, and focusing on the child's intuitive representations and informal procedures (Carey et al., 1995). This idea of a multicultural pedagogy receives further attention in Report No. 18 (Chapter 4, Section: *Children in Culturally Diverse Contexts*).

Socioeconomic Disadvantage

As noted in the introduction to this report, children in Ireland with low socioeconomic status perform less well, on average, than their more advantaged counterparts. Group differences are most notable among children attending schools in the urban dimension of the School Support Programme under DEIS, with average scores among children attending the most disadvantaged schools (those in urban DEIS Band 1) between three-fifths and four-fifths of a standard deviation below those of children in non-DEIS urban schools in the most recent national assessment (Eivers et al., 2010).

While the national assessments focus on the mathematical performance of children in second and sixth classes, international research indicates that the relationship between socioeconomic status (SES; usually defined in terms of parents' income level and/or education) and mathematics performance manifests itself considerably earlier (NRC, 2009). Pre-verbal number sense, which involves the ability to discriminate between large arrays of various sizes, begins in early infancy and appears to be universal (Xu, Spelke & Goddard, 2005). Preschool and early school number sense, which involves an understanding of number words and symbols, is more heavily influenced by experience and instruction, and large differences in performance are evident by the time children enter preschool, on standardised tests and on measures such as determining set size, comparing sets, or carrying out calculations (Klibanoff et al., 2006). Early differences in mathematical performance between children in families described as middle- and low-SES emerge for spatial/geometric understanding and measures as well as for number competencies (Clements, Sarama & Gerber, 2005). The importance of preschool number sense is underlined by strong correlations between measures of number sense at preschool level and success on mathematics later in childhood (e.g., Fuchs et al., 2007). Young children's number skills can be measured using either verbal tasks (i.e., number tasks without objects) or non-verbal tasks (i.e., number tasks with objects). Children in disadvantaged circumstances often perform less well on the former and their growth rates in kindergarten and first grade tend to be lower, compared with number tasks in which objects are present. Differences have also been observed on the same number tasks when presented verbally and non-verbally, with performance among children identified as disadvantaged being lower in verbal contexts (Jordan et al., 2007).

However, there is also evidence that early knowledge of number words can assist young children on tasks that do not require verbal input, such as matching arrays of visual dots. In a study by Ehrlich et al. (2006), children (aged 2–3 years) identified as low-SES tended to do less well on such tasks, compared with children identified as middle-SES, though differences were eliminated if responses that were plus or minus 1 from the correct answer were accepted as correct. This research suggests that preschool children identified as low-SES have approximations of set sizes and number words, at a time when other children have achieved exact representations.

The US NRC Report (NRC, 2009) suggests that early differences on mathematical tasks among children with differing backgrounds can arise for a number of reasons, including the amount of support for mathematics at home and language and contextual factors. Clements and Sarama (2008) also point out that preschool programmes in the US that serve the most disadvantaged children tend to provide fewer opportunities and support for mathematics development than programmes that support more advantaged children. Jordan and Levine (2009) take the view that weaknesses in number competence can be identified in early childhood and that most children (including the most disadvantaged) have the capacity to develop the number competence that lays the foundation for later learning.

There is a broad range of factors that need to be considered in designing programmes to support the early mathematical development of less-advantaged children. These include

- *Parental beliefs and behaviours*. Parents in general prioritise the development of early literacy and language skills and living skills over mathematical skills (Barbarin et al., 2008) and expect preschools to focus on language and literary skills rather than early number skills (Cannon & Ginsburg, 2008).
- *Nature of parent-child interactions*. Even in studies in which parents involve their less- or more-advantaged children in informal mathematical activities such as talking about number, playing with puzzles and shapes, engaging in counting, and using number symbols to represent quantity, differences in mathematical performance between children with varying degrees of disadvantage can arise, suggesting subtle differences in the effectiveness of parents in differing contexts (Saxe et al., 1987).
- *Language.* There is a wide variation in instances of mathematical language both in homes and in preschool settings (see Chapter 3, Section: *Variation in Language Skills and Impact on Mathematics*), which can impact on children's developing mathematical competence in a range of areas, including number and spatial awareness.
- *Parent expectations.* Studies show a tendency among parents to overestimate their children's mathematical competence in aspects such as cardinality (Fluck, Linnell & Holgate, 2005). This may limit the frequency and intensity of mathematical activities involving young children and their parents.

Differences that characterise the home and community backgrounds of children living in disadvantage underlines the importance of: (a) supporting parents of such children to enhance their children's mathematical competence through engagement in and discussion on a range of mathematics-related activities; (b) ensuring that preschool programmes, as well as building on child-led play and naturally occurring opportunities, include a strong numeracy component that includes opportunities for children to engage in planned activities with varying degrees of structure that expose them to mathematical ideas; and (c) providing opportunities for children in home, preschool and primary school settings to engage in language interactions with adults about important mathematical ideas and symbols, whether during structured play or storytime. Although these activities are often recommended for all children, their frequency and intensity may need to be raised in contexts in which large numbers of children from disadvantaged backgrounds meet together.

There is a range of intervention programmes in place in DEIS schools in Ireland for children who may be at risk of mathematical difficulties, including *Mathematics Recovery (Mata), Ready, Set, Go Maths,* and *Maths for Fun*. While these programmes incorporate many of the goals of effective early years interventions, we could not find any published evaluations of the effects of the programmes on young children's mathematical development in the Irish context. It would seem important to monitor the effects of these programmes, and, in particular, the extent to which skills acquired through early intervention are maintained and extended once children exit from the programmes.

Conclusion

Individual and/or group variation should be regarded as a strength of the educational system and the redeveloped mathematics curriculum needs to address learner variability. It is not that distinctive teaching approaches (or indeed distinctive curricula) are required but that mathematics teaching should address specific needs – including the needs of those who are exceptional because of a disability or talent, those who do not have English or Irish as a mother language or those coming from a disadvantaged background. The implications of this position are immense. In particular, there is a need to move away from over-reliance on data from group testing to inform policy and practice in the area and to supplement group data with other relevant assessment information. Furthermore, teachers need to be supported in the design, development and delivery of mathematics lessons that recognise and capitalise on learner variability. Some of the challenges inherent in this task are explored in Report No. 18 in the discussion on an equitable curriculum (see Chapter 4, Section: *Children in Culturally Diverse Contexts*).

The key messages arising from this chapter are as follows:

- The groups of individuals that often require particular attention in the teaching and learning of mathematics are 'exceptional' children (those with developmental disabilities or who are especially talented at mathematics), children who speak a first language other than English/Irish at home, and children living in disadvantage. In addressing their individual needs, the use of multi-tiered tasks, in which different levels of challenge are incorporated, is advocated.
- Mathematics 'for all' implies a pedagogy that is culturally sensitive and takes account of individuals' ways of interpreting and making sense of mathematics. In particular, norms-based testing can disadvantage certain groups. A diverse range of assessment procedures is required to identify those who experience learning difficulties in mathematics.
- Parents and educators need particular supports in constructing mathematically-interactive and rich environments for children aged 3–8 years. The intensity of the support will need to vary according to the needs of particular groups of children.

CHAPTER 8

Key Implications

The purpose of this report is to inform the redevelopment of the mathematics curriculum for children aged 3–8. In addressing this we focused on research related to the mathematical education of children aged 3–8 years. We drew on a broad range of relevant literature and research studies, particularly those published since the introduction of the current Primary School Mathematics Curriculum in 1999. In line with the research request, we focused on definitions of mathematics education, theoretical perspectives, the role of language and communication in learning mathematics, goals, stages of development, diversity and assessing and planning for progression.

The implications for curriculum development presented here are based on a view of mathematics as useful and as a way of thinking, seeing and organising the world, as well as being aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004). They are based on a view of all children as problem-solvers who can make sense of the world using mathematics, who engage in the processes of mathematization, and who develop productive dispositions towards mathematics.

Our implications are presented in a context in which there is a growing awareness of children's early mathematical knowledge and how it can be developed. Other important contextual factors include the multicultural nature of children's learning environments, the ever-growing presence of technology in all aspects of children's lives, concerns about children's mathematical achievements and attitudes, and an economy in which mathematical knowledge is increasingly valued.

The key implications arising from this review of research presented in this report are as follows:

- In the curriculum, a view of all children as having the capacity to engage with deep and challenging mathematical ideas and processes from birth should be presented. From this perspective, and in order to address on-going concerns about mathematics at school level, a curriculum for 3–8 year-old children is critical. This curriculum needs to take account of the different educational settings that children experience during these years.
- The curriculum should be developed on the basis of conversations amongst all educators, including those involved in the NCCA's consultative structures and processes, about the nature of mathematics and what it means for young children to engage in doing mathematics. These conversations should be informed by current research, as synthesised in this report and in Report No. 18, which presents a view of mathematics as a human activity that develops in response to everyday problems.

- The overall aim of the curriculum should be the development of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). As mathematization plays a central role in developing proficiency, the processes of mathematization should permeate all learning and teaching activities. These include connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising.
- The curriculum should foreground mathematics learning and development as being dependent on children's active participation in social and cultural experiences, while also recognising the role of internal processes. This perspective on learning provides a powerful theoretical framework for mathematics education for young children. Such a framework requires careful explication in the curriculum, and its implications for pedagogy should be clearly communicated.
- In line with the theoretical framework underpinning the curriculum, mathematical discourse (math talk) should be integral to the learning and teaching process. The curriculum should also promote the development of children's mathematical language in learning situations where mathematics development may not be the primary goal. Particular recognition should be given to providing intensive language support, including mathematical language, to children at risk of mathematical difficulties.
- The goal statements of the curriculum should be aligned with its underlying theory. An approach whereby processes are foregrounded but content areas are also specified is consistent with a participatory approach to mathematics learning and development. In the curriculum, general goals need to be broken down for planning, teaching and assessment purposes. Critical ideas indicating the shifts in mathematical reasoning required for the development of key concepts should be identified.
- Based on the research which indicates that teachers' understanding of developmental progressions (learning paths) can help them with planning, educators should have access to information on general learning paths for the different domains. Any specification of learning paths should be consistent with sociocultural perspectives, which recognise the paths as provisional, non-linear, not age-related and strongly connected to children's engagement in mathematically-rich activity. Account needs to be taken of this in curriculum materials. Particular attention should be given to the provision of examples of practice, which can facilitate children's progression in mathematical thinking.
- The curriculum should foreground formative assessment as the main approach for assessing young children's mathematical learning, with particular emphasis on children's exercise of agency and their growing identities as mathematicians. Digital technologies offer particular potential in relation to these aspects of development. The appropriate use of screening/diagnostic tests should be emphasised as should the limitations of the use of standardised tests with children in the age range 3–8 years. The curriculum should recognise the complex variety of language

backgrounds of a significant minority of young children and should seek to maximise their meaningful participation in assessment.

- A key tenet of the curriculum should be the principle of 'mathematics for all'. Central to this is the vision of a multicultural curriculum which values the many ways in which children make sense of mathematics. While there are some groups or individuals who need particular supports in order to enhance their engagement with mathematics, in general distinct curricula should not be advocated.
- Curriculum developments of the nature described above are strongly contingent on concomitant developments in pre-service and in-service education for educators at preschool and primary levels.

GLOSSARY

Glossary

Abstraction

an idea based on experiences but independent of any one experience (NRC, 2001, p. 110); mathematics is about increasingly being able to deal with ideas rather than events.

Adaptive reasoning/expertise

the capacity for logical thought, reflection, explanation, and justification.

Big ideas

the overarching concepts that are mathematically central and coherent, consistent with children's thinking and generative of future learning (Clements and Sarama, 2007, p. 463);

overarching concepts that connect numerous topics and applications (Baroody et al., 2006, p. 205).

Conceptual understanding

understanding of mathematical concepts, operations and relations.

Context

an event, issue or situation derived from reality, which is meaningful to the children or which they can imagine and which leads to using mathematical methods from their own experience. It provides concrete meaning and support for the relevant mathematical relations or operations. Situations might be drawn from everyday experiences such as bus rides, or shopping, and handling money (van den Heuvel-Panhuizen 2008, p. 243);

Culture

the totality of artefacts, rites, stories and customs shared in a given human social group (Ryan & Williams, 2007, p. 161).

Developmental progression

a sequence of levels of thinking (Clements and Sarama, 2007, p. 463).

Dynamic Geometric Software (DGS) programs

tools that can be used to construct and manipulate geometric objects and relations (e.g. Battista, 1998). They help children to develop rich mental models which help them to reason in increasingly sophisticated ways (Battista, 2001).

Embodied cognition

understanding situated in the body, in space and time, as well as socioculturally and historically (Ryan & Williams, 2007).

Embodiment

an idea or abstraction expressed or represented physically or concretely (Ryan & Williams, 2007). For example, young children can explore number operations on a floor number line, by moving themselves forward and back on the line. They often communicate and articulate their understandings and ideas by using actions and gestures instead of/as well as words.

Formal mathematical knowledge

knowledge that is school taught, largely represented in written form and frequently the result of deliberate efforts by children and teachers (Baroody et al., 2006, p. 189).

Hypothetical learning trajectory (HLT)

instructional sequences or potential developmental paths that serve to focus educators' attention on teaching children rather than on teaching a curriculum (Baroody et al., 2006, p. 206).

Informal mathematical knowledge

knowledge gleaned from everyday activities in what are not normally considered instructional settings such as home, playground, grocery store, family car. Such knowledge is usually represented verbally or nonverbally and often learned incidentally (Baroody et al., 2006, p. 189).

IRF

the teacher initiation–student response–teacher feedback (IRF sequence) is a form of classroom interaction commonly practiced in classroom discourse (Sinclair & Coulthard, 1975). The sequence is contrasted with a participation structure that allows for student-initiated negotiations.

Learning trajectory

description of children's thinking and learning of a specific mathematical domain and a conjectured route for that learning to follow through a set of instructional activities (Clements, 2008).

Mathematical model

a bridge between informal understanding and the abstraction of formal ideas. A model can for instance include materials, visual sketches or symbols. The models are formulated by children themselves in the course of their engagement with a problem (van den Heuvel-Panhuizen, 2003).

Mathematical processes

general mathematical processes such as problem-solving, reasoning and proof, communicating, connecting, and representing; justifying, argumentation; generalising;

mechanisms by which children can go back and forth between the abstract mathematics and real situations in the world around them (NRC, 2009, p. 43).

Mathematical proficiency

consists of the five intertwined and interrelated strands of conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (NRC, 2001).

Mathematizing

the analysing of real-world problems in a mathematical way (Treffers & Beishuizen, 1999, p. 32);

casting children's actions (work) into explicitly mathematical form (e.g. Ginsburg 2009b, p. 123);

to conceive of problems in explicitly mathematical terms (Ginsburg 2009a, p. 412);

formulating real situations in mathematical terms (NRC, 2009, p. 43; 354);

involves reinventing, re-describing, reorganising, quantifying, structuring, abstracting, generalising, and refining that which is first understood on an intuitive and informal level in the context of everyday activity (Clements, Sarama & DiBiase, 2004, p. 12);

…organising information into charts and tables, noticing and exploring patterns, putting forth explanations and conjectures, and trying to convince one another of their thinking (Fosnot & Dolk, 2001, pp. 4–5);

…more than process is happening. Children [can be] exploring ideas such as quantity and unitizing, and division, in relation to their own level of mathematical development. And mathematizing should not be dismissed as simply process. Mathematizing is content. (Fosnot & Dolk, 2001, p.9)

Model context

can stand for a whole range of related arithmetic situations in which the operations of addition, subtraction, multiplication and division are meaningfully reflected. It can provide support in enabling children to carry out a calculation or develop a procedure (van den Heuvel-Panhuizen, 2008, p. 91).

Modeling problems

can be contrasted with traditional 'word' problems since the information given is often in a form (for example, a table) that must be interpreted by the child. Problems revolve around authentic situations that need to be interpreted and described in mathematical ways. (English & Watters, 2004)

Modelling

a process through which children learn how to behave as mathematicians by imitating (modelling) the behaviour of others. Adults can teach children how to act mathematically by presenting them with examples of the dispositions, attitudes and values which the adults around them consider to be appropriate. Modelling occurs when children internalise these behaviours
(Adapted from MacNaughton & Williams, 2004).

Procedural fluency

skill in carrying out procedures flexibly, accurately, efficiently and appropriately (NRC, 2001).

Productive disposition

the habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and in one's own efficacy (NRC, 2001).

Reflective abstraction

for Piagetians, reflective abstraction is a key process for activating accommodation and assimilation, or restructuring one's schema/models. This implies that children learn from talking about and reflecting on their mathematical ideas and solutions/strategies with others (Ryan & Williams, 2007, p. 158).

RME

is an acronym for Realistic Mathematics Education – an approach to mathematics education devised by Freudenthal in the Netherlands in the 1970s (see *Chapter 5*).

Routine expertise

mastery of basic skill and other skills by rote (Baroody et al., 2006, p. 2001).

Self-regulation

where the learner takes control and ownership of their own learning.

Specific language impairment (SLI)

a language disorder that delays the mastery of language skills in children who have no hearing loss or other developmental delays. SLI is also called developmental language disorder, language delay, or developmental dysphasia. It is one of the most common childhood learning disabilities, affecting approximately 7 to 8 percent of children in kindergarten. The impact of SLI persists into adulthood. Definition taken from *https://www.nidcd.nih.gov/health/voice/pages/specific-language-impairment.aspx*

Strategic competence

the ability to formulate, represent, and solve mathematical problems (NRC, 2001).

Working memory

relates to the task at hand, and coordinates the recall of memories necessary to complete it.

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ISSN 1649-3362

National Council for Curriculum and Assessment
35 Fitzwilliam Square, Dublin 2.

T: +353 1 661 7177
F: +353 1 661 7180
E: info@ncca.ie

www.ncca.ie

Design by www.principle.ie

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An Chomhairle Náisiúnta Curaclaim agus Measúnachta
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Research Report No. 18

Mathematics in Early Childhood and Primary Education (3-8 years)

Teaching and Learning

Thérèse Dooley, Elizabeth Dunphy and Gerry Shiel

With Deirdre Butler, Dolores Corcoran, Thérèse Farrell, Siún NicMhuirí, Maura O'Connor, and Joe Travers

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National Council for Curriculum and Assessment
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T: +353 1 661 7177
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E: info@ncca.ie

www.ncca.ie

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Thérèse Dooley, St. Patrick's College, Dublin

Elizabeth Dunphy, St. Patrick's College, Dublin

Gerry Shiel, Educational Research Centre, Dublin

Deirdre Butler, St. Patrick's College, Dublin

Dolores Corcoran, St. Patrick's College, Dublin

Thérèse Farrell, St. Patrick's College, Dublin

Siún NicMhuirí, St. Patrick's College, Dublin

Maura O'Connor, St. Patrick's College, Dublin

Joe Travers, St. Patrick's College, Dublin

Professor Bob Perry, Charles Sturt University, Australia

Charles Sturt University

Educational Research Centre
Foras Taighde Ar Oideachas

Research conducted on behalf of the
National Council for Curriculum and Assessment

2014

Acronyms

AAMT	Australian Association of Mathematics Teachers
Aistear	The Early Childhood Curriculum Framework (2009)
ACARA	Australian Curriculum, Assessment and Reporting Authority
AfA	Achievement for All (UK intervention)
ASD	Autistic Spectrum Disorders
CCEA	Council for Curriculum, Examinations and Assessment (Northern Ireland)
CCK	Common Content Knowledge
CCSSM	Common Core States Standards for Mathematics (United States)
COMET	Cases of Mathematics Instruction to Enhance Teaching (Silver et al., 2007)
CPD	Continuing Professional Development
DEIS	Delivering Equality of Opportunities in Schools
DES	Department of Education and Skills (formerly Department of Education and Science)
ECA	Early Childhood Australia
ECCE	Early Childhood Care and Education
EMS	European Mathematical Society
HSCL	Home School Community Liaison (Ireland)
ICT	Information and Communications Technology
IWB	Interactive White Board
KCC	Knowledge of Content and Curriculum
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
KQ	Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005)
MKT	Mathematical Knowledge for Teaching
NAEYC	National Association for the Education of Young Children (United States)
NCCA	National Council for Curriculum and Assessment
NCTM	National Council of Teachers of Mathematics (United States)
NRC	National Research Council (United States)
PCK	Pedagogical Content Knowledge
PSMC	Primary School Mathematics Curriculum (1999)
PUFM	Profound Understanding of Fundamental Mathematics (Ma,1999)
REPEY	Researching Effective Pedagogy in the Early Years (Siraj-Blatchford et al., 2002)
RME	Realistic Mathematics Education
SCK	Specialised Content Knowledge
SEND	Special Educational Needs and Disabilities
SES	Socioeconomic status
SKIMA	Subject Knowledge in Mathematics (Rowland, Martyn, Barber, & Heal, 2001)
SSP	School Support Programme
TAL	Tussendoelen Annex Leerlijinen (In Dutch); Intermediate Attainment Targets (in English)
TIMSS	Trends in International Mathematics and Science Study

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The executive summaries of reports No. 17 and No. 18 are available online at *ncca.ie/primarymaths*. The online versions include some hyperlinks which appear as text on dotted lines in this print copy.

Acknowledgements

The authors thank the National Council for Curriculum and Assessment for commissioning and supporting this report. They are very thankful to Arlene Forster and Aoife Kelly of the NCCA for providing detailed feedback on earlier drafts of the report. They are also indebted to Professor Bob Perry, Charles Sturt University Australia who read early drafts of the report and who provided expert advice on various issues addressed in the report.

Executive Summary

The review of research on mathematics learning of children aged 3–8 years is presented in two reports. These are part of the NCCA's Research Report Series (ISSN 1649–3362). The first report (Research Report No. 17) focuses on theoretical aspects underpinning the development of mathematics education for young children. The second report (Research Report No. 18) is concerned with related pedagogical implications. The key messages from Report No. 18 are presented in this Executive Summary.

A View of Mathematics

Both volumes are underpinned by a view of mathematics espoused by Hersh (1997): mathematics as 'a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context' (p. xi). Mathematics is viewed not only as useful and as a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004). All children are viewed as having an ability to solve mathematical problems, make sense of the world using mathematics, and communicate their mathematical thinking. This shift in perspective demands a change in pedagogy – in particular it puts the teaching-learning relationship at the heart of mathematics.

Context

In Report No. 17 we argue that the overall aim of the curriculum should be the development of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) (National Research Council [NRC], 2001). As mathematization plays a central role in developing proficiency, the processes of mathematization should permeate all learning and teaching activities. These include connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. Foregrounding mathematical proficiency as the aim of mathematics education has the potential to change the kind of mathematics and mathematical learning that young children experience. As it demands significant changes in pedagogy, curriculum and curricular supports (Anthony & Walshaw, 2007), it also poses challenges that are wide-ranging and systemic.

The development of mathematical proficiency begins in the preschool years, and individuals become increasingly mathematically proficient over their years in educational settings. This implies that educators in the range of early childhood settings need to develop effective pedagogical practices that engage learners in high-quality mathematics experiences. There is a concomitant need to address issues related to curriculum content and presentation. In particular, the questions of how to develop a coherent curriculum and how to formulate progressions in key aspects of mathematics are important. The view of curriculum presented in this report is both wide and dynamic. It is recognised that the mathematics education of young children extends beyond the walls of the classroom: family and the wider community can make a significant contribution to children's mathematical achievement (e.g., Sheldon & Epstein, 2005).

Pedagogy

It is impossible to think about good mathematics pedagogy for children aged 3–8 years without acknowledging that much early mathematical learning occurs in the context of children's play (e.g. Seo & Ginsburg, 2004). Educators need to understand how mathematics learning is promoted by young children's engagement in play, and how best they can support that learning. For instance, adults can help children to maximise their learning by helping them to represent and reflect on their experiences (e.g., Perry & Dockett, 2007a). Learning through play is seen as fundamental to good mathematics pedagogy in early childhood. It assumes varying degrees of emphasis depending on the age of the child. Recent research points to a number of other important principles which underpin good mathematics pedagogy for children aged 3–8 years (e.g., Anthony & Walshaw, 2009a; NRC, 2005). These principles focus on people and relationships, the learning environment and learners. Features of good mathematics pedagogy can be identified with reference to these principles. Both the principles and the features of pedagogy are consistent with the aim of helping children to develop mathematical proficiency. They pertain to all early educational settings, and are important in promoting continuity in pedagogical approaches across settings.

Practices

Good mathematics pedagogy incorporates a number of meta-practices (i.e., overarching practices) including the promotion of math talk, the development of a productive disposition, an emphasis on mathematical modeling, the use of cognitively challenging tasks, and formative assessment. The literature offers a range of perspectives, and advice, as to the issues for educators in integrating these elements into their practices. In doing so, the vision of 'mathematics for all' is supported.

Good mathematics pedagogy can be enacted when educators engage children in a variety of mathematically-related activities across different areas of learning. The activities should arise from children's interests, questions, concerns and everyday experiences. A deep understanding of the features of good pedagogy should inform the ways in which educators engage children in mathematically-related activities such as play, story/picture-book reading, project work, the arts and physical education. The potential of these activities for developing mathematical proficiency can best be realised when educators focus on children's mathematical sense-making. In addition, educators need to maximise the opportunities afforded by a range of tools, including digital tools, to mediate learning.

Curriculum Development

Goals, coherent with the aim of mathematical proficiency, should be identified. These goals relate both to process and content. The processes of mathematization, that is, communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting, should be foregrounded. In line with the principle of 'mathematics for all', each of the five content domains – Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance – should be given appropriate attention.

Goals need to be broken down for planning, teaching and assessment purposes. Learning paths can be helpful for this purpose. As is outlined in Report No. 17, differences in the ways learning paths are presented in the literature rest largely on their theoretical underpinnings. For example, developmental progressions described by Sarama and Clements (2009) are finely grained and age-related, whereas the TAL[1] trajectories developed in the context of Realistic Mathematics Education (van den Heuvel-Panhuizen, 2008) are characterised by fluidity and the role of context. In line with a sociocultural approach to the learning of mathematics, we advocate that learning paths be used in a flexible way to posit shifts in mathematical reasoning, i.e. critical ideas in each of the domains. Narrative descriptors of critical ideas can be used to inform planning and assessment. Learning outcomes, relating to content domains and processes, can then be derived from a consideration of the goals, learning paths and narrative descriptors. The figure below shows an emerging curriculum model highlighting how the relationships between the different elements may be conceptualised.

1 In Dutch, learning-teaching trajectories are referred to as TALs (i.e., Tussendoelen Annex Leerlijinen).

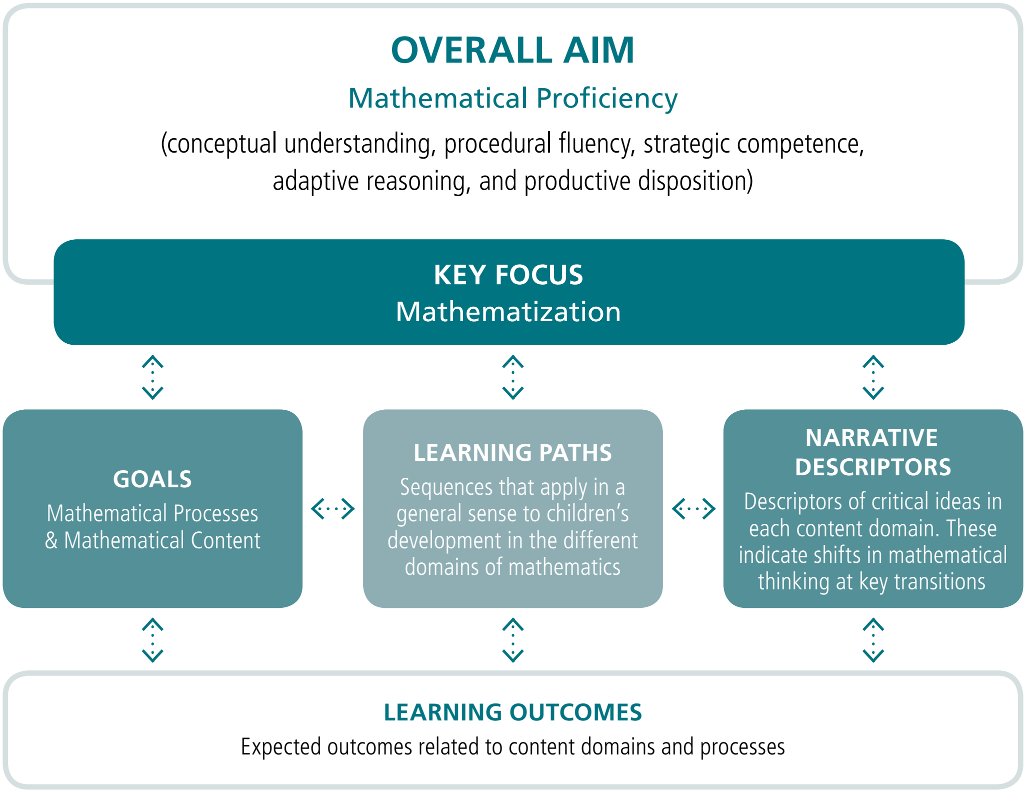

Figure ES.1: Emerging Curriculum Model

Curricular Issues

While the specification of processes and content in the mathematics curriculum is critically important, attention should also be given to issues that relate to curriculum access and curriculum implementation. This is based on the premise that the curriculum must serve all children, including exceptional children (those with developmental delays and those with exceptional talent) and children in culturally diverse contexts. Other key issues include the timing of early intervention, the allocation of time to mathematics in early learning settings, and how best to achieve the integration of mathematics across the curriculum.

Consistent with Lewis and Norwich's (2005) concept of continua of common teaching approaches that can be subject to varying degrees of intensity depending on children's needs, modifications to the mathematics curriculum for children with special education needs are proposed. Mathematically-talented children should be supported in deepening their understanding of and engagement with the existing curriculum rather than being provided with an alternative one. In the case of English-language learners, and children attending Irish-medium schools, the key role of mathematical discourse and associated strategies in enabling access to the language in which the curriculum is taught are emphasised (e.g., Chapin, O'Connor, & Anderson, 2009). Attention to language is also highlighted as a critical issue in raising the mathematics achievement of children in DEIS schools. More generally, it is noted that there is now strong research indicating that additional support should be provided at an earlier stage than is indicated in current policy documents (e.g. Dowker, 2004; 2009). There is a need to allocate sustained time to mathematics to ensure that all children engage in mathematization. Dedicated and integrated time provision is recommended. The value of integrating mathematics across areas of learning is recognised, though it is acknowledged that relatively little research is available on how best to achieve this.

Partnership with Parents

In line with the emphasis on parental involvement in the *National Strategy to Improve Literacy and Numeracy among Children and Young People (2011–2020)* (Department of Education and Skills [DES], 2011a), the key role of parents in supporting children to engage in mathematics is emphasised. There is a range of activities in which parents can engage with schools so that both parents and educators better understand children's mathematics learning. However, it is acknowledged that research on parental involvement in mathematics lags behind similar research relating to parental involvement in reading literacy.

In the literature on parental involvement, the need to establish a continuous, two-way flow of information about children's mathematics learning between educators and parents is a key theme. There is potential for technology to support this. Strategies designed to support parents to better understand their child's mathematical learning include observation of and discussion on children's engagement in mathematical activities in education settings. Mechanisms are required to inform parents about the importance of mathematics learning in the early years, and what constitutes mathematical activity and learning for young children. The significant role that parents play in the mathematical development of their children should be foregrounded.

Teacher Preparation and Development

Curriculum redevelopment is strongly contingent on parallel developments in pre-service and in-service education for educators across the range of settings. In particular, professional development programmes need to focus on the features of good mathematics pedagogy and the important meta-practices that arise from these.

In order for teachers to foster mathematical proficiency in children, they themselves need to be mathematically proficient. Therefore, teacher preparation courses need to provide opportunities for pre-service teachers to engage in rich mathematical tasks. Educators need to develop mathematical knowledge for teaching through a collaborative focus on teaching and learning of mathematics. They need opportunities to notice children's engagement in mathematics and responses to mathematical ideas. Case studies of practice are valuable tools in this regard. These can be used by pre-service (and in-service) teachers to question and critique the practice of others in order to develop 'local knowledge of practice' (Cochran-Smith, 2012, p. 46).

Among the recommendations for the continuing professional development of teachers (CPD) is investment in stronger systems of clinical supervision across the preparation-induction boundary (Grossman, 2010). The notion of clinical supervision could mean an emphasis on developing good mathematics teaching practices through collaborative review and reflection on existing practice. This is important because inquiry as a stance has been advocated as a successful key to teacher change (Jaworski, 2006). In this regard, lesson study is a practice that is currently foregrounded in the literature as a significant development in school-based professional development (e.g., Corcoran & Pepperell, 2011; Fernández, 2005). In lesson study, publicly available records of practice or 'actionable artifacts' are important by-products (Lewis, Perry, & Murata, 2006, p. 6). The practice offers opportunities at school and classroom level for enactment of critical inquiry into mathematics lessons.

Key Implications

The key implications for the redevelopment of the mathematics curriculum arising from the review of research presented in this report are as follows:

- The curriculum should be coherent in terms of aims, goals relating to both processes and content, and pedagogy. (*Chapter 1*, *Chapter 3*)
- The processes of mathematization, that is, communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting, should be foregrounded in curriculum documentation and should be central to the mathematical experiences of all children. (*Chapter 2*, *Chapter 3*)
- The redeveloped mathematics curriculum needs to acknowledge and build on the pedagogical emphases in *Aistear*. (*Chapter 2*)
- In order to facilitate transitions, educators across early education settings need to communicate about children's mathematical experiences and the features of pedagogy that support children's learning. (*Chapter 1*)
- The principles and features of good mathematics pedagogy as they pertain to people and relationships, the learning environment, and the learner, should be emphasised. (*Chapter 1*)
- The overarching meta-practices and the ways in which they permeate learning activities should be clearly explicated. (*Chapter 2*)

- Educators should be supported in the design and development of rich and challenging mathematical tasks that are appropriate to their children's learning needs. (*Chapter 2*, *Chapter 5*)
- The curriculum should exemplify how tools, including digital tools, can enhance mathematics learning. (*Chapter 2*)
- Children should engage with all five content domains – Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance. The strand of Early Mathematical Activities as presented in the current PSMC should be integrated into the five content areas. (*Chapter 3*)
- In curriculum documentation, critical ideas in each content domain need to be explicated and expressed as narrative descriptors. These critical ideas, derived from learning paths, should serve as reference points for planning and assessment. In presenting these ideas, over-specification should be avoided. Learning outcomes arising from these also need to be articulated. (*Chapter 3*)
- Narrative descriptors of mathematical development, that is, descriptions of critical ideas, should be developed in class bands, e.g., two years. These critical ideas indicate shifts in children's mathematical reasoning in each of the content domains. (*Chapter 3*)
- The principles of equity and access should underpin the redeveloped mathematics curriculum. The nature of support that enables exceptional children (those with developmental delays and those with exceptional talent), children in culturally diverse contexts and children in disadvantaged circumstances to experience rich and engaging mathematics should be specified. (*Chapter 4*)
- Additional support/intervention for children at risk of mathematical difficulties should begin at a much earlier point than is specified in the current guidelines. (*Chapter 4*)
- Learning outcomes in mathematics should be cross-referenced with other areas of learning and vice-versa, in order to facilitate integration across the curriculum. (*Chapter 2*, *Chapter 4*)
- Additional time allocated for mathematics should reflect the increased emphases on mathematization and its associated processes. Some of this additional time might result from integration of mathematics across areas of learning. While integration has the potential to develop deep mathematical understanding, the challenges that it poses to teachers must be recognised. (*Chapter 3*, *Chapter 4*)
- Ongoing communication and dialogue with parents and the wider community should focus on the importance of mathematics learning in the early years, the goals of the mathematics curriculum and ways in which children can be supported to achieve these goals. (*Chapter 5*)
- Structures should be put in place that encourage and enable the development of mathematical knowledge for pre-service and in-service teachers. Educators need to be informed about goals, learning paths and critical ideas. Records of practice, to be used as a basis for inquiry into children's mathematical learning and thinking, need to be developed. (*Chapter 6*)

- Educators need to be given opportunities to interrogate and negotiate the redeveloped curriculum with colleagues as it relates to their setting and context. Time needs to be made available to educators to engage in collaborative practices such as lesson study. (*Chapter 6*)
- Given the complexities involved, it is imperative that all educators of children aged 3–8 years develop the knowledge, skills, and dispositions required to teach mathematics well. (*Chapter 6*)
- Given the central importance of mathematics learning in early childhood and as a foundation for later development, mathematics should be accorded a high priority, at both policy and school levels, similar to that accorded to literacy. (*Chapter 4*, *Chapter 5*)

Introduction

In Report No. 17, we identified a number of implications for mathematics pedagogy and curriculum for children aged 3–8 years that arise from the research literature. These include the following:

- The curriculum should present a view of all children as having the capacity to engage with deep and challenging mathematical ideas and processes from birth.
- The overall aim of the curriculum should be the development of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). As mathematization plays a central role in developing proficiency, the processes of mathematization should permeate all learning and teaching activities. These include connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising.
- The curriculum should foreground mathematics learning and development as dependent on children's active participation in social and cultural experiences, while also recognising the role of internal processes.
- In line with the theoretical framework underpinning the proposed curriculum, mathematical discourse (math talk) should be integral to the learning and teaching process.
- The goal statements of the curriculum should be aligned with its underlying theory. An approach whereby processes are foregrounded, but content areas are also specified, is consistent with a participatory approach to mathematics learning and development. In the curriculum, general goals need to be broken down for planning, teaching and assessment purposes. Critical ideas indicating the shifts in mathematical reasoning required for the development of key concepts should be identified.
- Based on the research which indicates that teachers' understanding of developmental progressions (learning paths) can help them with planning, educators should have access to information on general learning paths for the different domains.
- The curriculum should foreground formative assessment as the main approach for assessing young children's mathematical learning, with particular emphasis on children's exercise of agency and their growing identities as mathematicians.
- A key tenet of the curriculum should be 'mathematics for all'. Central to this is the vision of a multicultural curriculum which values the many ways in which children make sense of mathematics.

This report explores these implications. As in Report No. 17, it is premised on a view of mathematics as not only useful and a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen et al., 2004). It recognises that mathematics is 'power', often acting as a 'gatekeeper to social success' (Gates & Vistro-Yu, 2003, p. 32), in that individuals who do well in the subject are likely to have better access to jobs, college courses and higher incomes than those who do not (Dooley & Corcoran, 2007). It is also premised on the view that mathematics is not absolute and certain but is constructed by a community of learners. In the words of Hersh (1997, p. xi), mathematics is 'a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context'. This report is thus also founded on a view of all individuals having an innate ability to solve problems and make sense of the world. This shift in perspective demands a change in pedagogy – in particular it puts the teaching-learning relationship at the heart of mathematics.

While the view of mathematics as a problem-solving activity permeates the 1999 Primary School Mathematics Curriculum (PSMC) (Government of Ireland, 1999a), there is considerable evidence of a mismatch between this view and the mathematics that the majority of children experience in the classroom. In a survey conducted by the Inspectorate of the then Department of Education and Science (DES) (DES, 2005a) of 61 classes, it was found that there was an over-reliance on traditional textbook problems in almost one third of classes. National surveys of mathematics confirm this situation – indeed textbooks are used by most primary school children on a daily basis and they act as the main planning tool for the majority of teachers in second, fourth and sixth classes (Eivers, et al., 2010; Shiel, Surgenor, Close, & Millar, 2006). Moreover, findings of studies conducted by Dunphy (2009) and Murphy (2004) suggest a strict adherence to textbook activities by many teachers of junior infant and senior infant classes respectively. The continued emphasis on lower-order thinking skills and mathematical procedures in Irish schools means that many children are being denied the opportunity to experience mathematics as a creative and engaging process. Ford and Harris (1992) suggest that the creativity that is innate to young preschool children is inhibited by a school system that rewards convergent thinking and approves correct answers. Barnes (2000) contends that students need to experience excitement and work with uncertainty in order for moments of insight to occur and suggests that

> *[I]f instruction progresses by small, simple steps, and the teacher anticipates difficulties and provides immediate clarification, students will have less need to struggle and less occasion to make efforts of their own to achieve understanding and insight.* (p. 41)

Sinclair and Watson (2001) make a similar argument:

> *Learners may need to be inducted into the wonder of mathematics, to experience wonder vicariously through the teacher (including the stages of pleasure and frustration that sense-making requires) and, more urgently, to set aside the illusion of mathematics as systematic knowledge so complete that there is nothing more to expect.* (p. 41)

While the worked examples that tend to predominate textbook pages may serve a purpose in promoting procedural fluency, they are generally not helpful in providing children with opportunities to develop the other strands of mathematical proficiency and to experience the excitement and wonder of mathematics.

Foregrounding mathematical proficiency as the aim of mathematics education has the potential to change the kind of mathematics and mathematical learning that young children experience. It also poses challenges that are wide-ranging and systemic. Anthony and Walshaw (2007), in their 'best evidence synthesis' about what pedagogical approaches work to improve children's learning of mathematics, say of the mathematical proficiency strands:

> *These are the characteristics of an apprentice user and maker of mathematics and are appropriated by the student through effective classroom processes. They incorporate curriculum content, classroom organisational structures, instructional and assessment strategies, and classroom discourse regarding what mathematics is, how and why it is to be learned and who can learn it.* (p. 5)

It follows that changes in the mathematics experienced by young children demand significant changes in pedagogy, curriculum and curricular supports, each of which is addressed in this volume.

Pedagogy

A decade or so ago it was observed that we just didn't know enough about the issue of effectively supporting early mathematics learning across the age range 3–8 years (e.g. Gifford 2004; Ginsburg & Golbeck, 2004). It is still the case that mathematics teaching and learning in the prior-to-school years is under-researched (e.g., Anthony & Walshaw, 2009a). In Report No. 17 we identified mathematical proficiency as a key aim of mathematics education. The strands are interwoven and interdependent and each of the strands becomes progressively more developed in children as their mathematical experiences become increasingly sophisticated. As children develop proficiency in one strand, there are developments in other strands also. As we understand it, the development of mathematical proficiency begins in the preschool years, and individuals become increasingly mathematically proficient over their years in educational settings. As pointed out above, the strands develop as a result of good pedagogy. In this volume, the key issues related to good pedagogy are examined.

In *Chapter 1* a number of important principles which underpin the features of good mathematics pedagogy for young children are identified. These principles focus on people and relationships, the learning environment, and the learner. Arising from these principles, we present features of good mathematics pedagogy.

We survey the literature on key overarching pedagogical practices – math talk, disposition, modeling, tasks and assessment – in *Chapter 2*. We also give attention to a variety of mathematically-related activities across different areas of learning. We argue that these activities should arise from children's interests, questions, concerns and everyday experiences. Furthermore, educators need to focus on children's mathematical sense-making so that the strands of mathematical proficiency are developed.

Curriculum

It is now widely acknowledged that along with addressing pedagogy, there is a concomitant need to address issues related to curriculum content and presentation. In terms of curriculum redevelopment in Ireland, the question of how to formulate progressions in key aspects of mathematics arises. Some observations can be made about the presentation of content in the 1999 PSMC (Government of Ireland, 1999a). Here we find a large number of content objectives for each class level, e.g., 55 objectives for first class and 60 for second class (Murata, 2011). This specificity is reflected across all years, including the most junior classes. Murata argues that while this may be helpful in supporting educators to make connections across classes, it also carries the risk that educators see learning of mathematics as made up of mastery of discrete units without connections. Furthermore, the listing of the objectives as a list of competencies (e.g., all content objectives are preceded by the phrase, 'The child should be enabled to …') results in a reduction of content to basics and in frequent testing to decide on individual children's levels of mastery. In this volume, we consider ways in which the mathematics curriculum for 3–8 year olds might be redeveloped and represented.

In *Chapter 3*, we address the development of a coherent curriculum, where there is close alignment between the aim of mathematical proficiency and goals related to processes and content. We describe each of the processes associated with mathematization. We identify key issues in content areas related to Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance. We also examine how curricula in other jurisdictions are presented.

We draw attention to some important curricular issues in *Chapter 4*. In particular, we explore the idea of an equitable curriculum to which all children, including exceptional children and children in culturally diverse contexts, have access. In order that all children have access to powerful mathematical ideas, there are other questions that need to be addressed. Among them are: the timing of early intervention in mathematics, the allocation of time to mathematics in early learning settings, and the integration of mathematics across the curriculum. We present some findings on how such questions might best be addressed.

Curricular Supports

Remillard (1999) suggests that there are two levels to curriculum development – one is the conceptualisation of plans and the development of resources for teachers; the second is what teachers 'do' to implement these plans in their classrooms. A teacher, therefore, is integral to curriculum development. It is also recognised that the mathematics education of young children extends beyond the walls of the classroom – family and the wider community can make a significant contribution to children's mathematical achievement (e.g., Sheldon & Epstein, 2005). Therefore attention is also given in this report to the ways that teachers and parents can be supported to cultivate a rich mathematical learning environment for 3- to 8-year-old learners.

In *Chapter 5* we look at the importance of parents engaging in discussion with their child about mathematically-related activities that arise in the home, and in the context of homework when appropriate. Collaboration and sharing of information between parents and teachers are highlighted. Reference is made to various initiatives involving parents and local communities.

In *Chapter 6*, we explore issues related to teacher preparation and development. If pre-service and in-service educators of young children are to promote good mathematics learning, they must have a strong working knowledge of mathematics, and an openness to and facility with the processes of mathematization. We examine the construct of Mathematical Knowledge for Teaching (MKT) which is integral to teacher preparation and development. The need to develop MKT through critical and collaborative inquiry is addressed and different models of professional development are described.

In *Chapter 7*, we outline key implications for the redevelopment of the PSMC for children for 3–8 years of age arising from this volume.

CHAPTER 1

Good Mathematics Pedagogy

Pedagogy has been defined as '…the deliberate process of cultivating development' (Bowman, Donovan, & Burns, 2001, p. 182). A high degree of direct adult engagement and strong guidance is implicated in this definition, and such engagement is particularly necessary in relation to mathematics learning and teaching (e.g., Ginsburg et al., 2005). In pedagogical terms, the educator engages in practices that promote and assess early mathematics learning. Siraj-Blatchford, Sylva, Muttock, Gilden & Bell, (2002, p. 27) describe pedagogy as 'the practice (or the art, the science or the craft) of teaching' but they also point out that any adequate conception of educational practices for young children must be wide enough to include the provision of learning environments for play and exploration.

Starting with Play

Children's play and interests are the foundation of their first mathematical experiences. Play and playful activities/situations provide the main contexts in which most of children's prior- to-school mathematics learning takes place (e.g., Seo & Ginsburg, 2004; Van Oers, 2010). We know that children in their free play spontaneously engage in a great deal of mathematics, some of it at levels that are quite advanced. Sometimes they may play with mathematics itself (e.g., Ginsburg, Inoue, & Seo, 1999). Child-initiated play is central to the activity of young children and much mathematical learning occurs within the play environment (Montague-Smith & Price, 2012; Moyles, 2005). These play experiences become mathematical as children represent and reflect on them (Sarama & Clements, 2009).

Play provides a context wherein children can reflect on their past experiences, make connections across experiences, represent these experiences in different ways, explore possibilities and create meaning. These processes of play have strong connections to mathematical thinking (Perry & Dockett, 2007a). Play is a rich context for the promotion of mathematical language and concepts.

The adults around the child are often unaware of the child's engagement with mathematical ideas, and may not generally recognise this engagement and how it arises from children's spontaneous interests in, and exploration of, the world around them (Ginsburg, 2009a). Play is a context within which children can explore their mathematical ideas but it also provides a context within which adults can support and develop children's ideas. The adults with whom children interact have a critical role in helping children to reflect on (and talk about) their experiences in play and so to maximise the learning potential. Sensitive structuring of children's play can be effective in promoting mathematical thinking and learning (e.g., Ginsburg, et. al, 1999; Perry & Dockett, 2007a; Pound, 2008). From this perspective, learning through play is seen as fundamental to good mathematics pedagogy in early childhood. It assumes varying degrees of emphasis depending on the age of the child.

The potential of play to provide a 'bridging tool to school' is very significant (Broström, 2005). In recognising this potential, teachers need to integrate mathematics learning within children's play activity. For instance, the incremental development of children's spatial-geometric reasoning and their geometric and measurement skills across the transition period can be achieved through a systematic approach to the teaching of related concepts. This approach allows for the integration of problem-solving skills and content knowledge (Casey, 2009). Play with blocks provides the context within which teachers can teach the key aspects of spatial reasoning. Children's early experiences with blocks includes open-ended play, but over a period of time teacher-guided activities can serve to focus the children on sequenced spatial problems. As children's experiences with the blocks grow, and as they engage in various problem-solving activities, initial concepts are strengthened. These are then extended in later activities. As children solve mathematically-related problems they should be encouraged to use a range of informal approaches and problem-solving strategies with the intention of guiding them, as their understanding increases, towards the most effective strategies; they should be encouraged to talk about and compare their strategies with those used by others and learning experiences should target critical ideas (i.e. where conceptual shifts are required) (Fuson, Kalchman, & Bransford, 2005).

As stated in Report No. 17 (see Chapter 1, Section: *Defining Mathematics Education for Children Aged 3–8 years*), there are now some key sources that educators can look to for guidance in relation to teaching practices. These include statements from *The National Association of Educators of Young Children* (NAEYC) in the United States, who joined forces with the *National Council of Teachers of Mathematics* (NCTM) to issue a position paper (NAEYC/NCTM, 2002/2010) on early childhood mathematics. Similarly, in Australia, *Early Childhood Australia* and the *Australian Association of Mathematics Teachers* set out their position on pedagogical practices for early childhood mathematics (AAMT/ECA, 2006). These two sets of recommendations share a concern to engage children in appropriate and sensitive ways in what are to them interesting, meaningful, challenging and worthwhile mathematical experiences. They promote a pedagogy which is interactive, engaging and supportive of all children's learning. They also provide an overarching framework for considering what is important in early mathematics pedagogy across early education settings.

In this chapter we explore the features of good mathematics pedagogy. But in order to derive these we first examine the principles underpinning good mathematics pedagogy. We consider two sets of principles, both of which explicitly place mathematical proficiency at the core of mathematics education. The first set of principles arises from the work of Anthony and Walshaw (2007; 2009a) from New Zealand, who synthesised available international research on effective pedagogy in mathematics. These principles emphasise people, relationships, and the learning environment. The second set of principles is from the United States and relates to the learner. It arises from the NRC Report (2005) which focused on how principles of learning can be applied to mathematics education. These sets of principles present two viewpoints on how mathematical proficiency can be supported by educators, and, in the discussion which follows, we bring them together to offer a comprehensive account of the features of good mathematics pedagogy.

Principles that Emphasise People, Relationships and the Learning Environment

Anthony and Walshaw (2007; 2009a) draw together available evidence about what pedagogical practices work to improve children's outcomes in mathematics. In doing so, they stress the importance of interrelationships, and the development of community in the classroom.[2] Arising from their research, they present the following principles that characterise effective settings and effective educators:

- an acknowledgement that all children, irrespective of age, have the capacity to become powerful mathematics learners
- a commitment to maximise access to mathematics
- empowerment of all to develop positive mathematical identities and knowledge
- holistic development for productive citizenship through mathematics
- relationships and the connectedness of both people and ideas
- interpersonal respect and sensitivity
- fairness and consistency (Anthony & Walshaw, 2007, p. 1).

Cobb (2007) observes that the image of effective pedagogy that emerges from Anthony and Walshaw's synthesis is that of teaching as a coherent system rather than a set of discrete, interchangeable strategies. This pedagogical system encompasses four elements that work together as a set of connected parts:

- a non-threatening classroom atmosphere
- instructional tasks
- tools and representations
- classroom discourse.

2 In this report we use the term classroom similar to the way in which it is used in the NAEYC/NCTM position statement (2002/2012), i.e. as referring to any group setting for 3- to 8-year-old children (e.g. preschool, family child care, primary school).

Anthony and Walshaw identify a range of features of effective pedagogy based on people, relationships and the learning environment. These features are seen to enhance the development of young children's mathematical identities, dispositions and competencies. For instance, they include a balance between teacher-directed/initiated and child-directed/initiated activity and a focus on appropriate relationships. Everyday activity, including play, is seen to provide a rich context for learning but, as they observe, 'unstructured play, by itself, is unlikely to provide sufficient support for young children's mathematical development' (p. 110). Their findings indicate that providing for children's optimum development through their access to explicitly mathematical experiences, and for their engagement in interactions which support and extend their mathematics learning, are both critical dimensions of the learning environment. The development of an increased focus on mathematical activities, games, books and technology are some of the experiences that are seen to enhance opportunities for learning. The research indicates that educators' increased mathematical awareness enables them to recognise and respond to opportunities for developing all children's ideas and for enhancing mathematics learning. The importance of responsive pedagogical interactions by adults with children is foregrounded, as is the engagement of children in discussions which promote their abilities to express their thinking and to conjecture, predict and verify. Differences in home experiences of children, with some families more orientated to mathematics than others, is seen as an issue of which educators need to be aware. The necessity of acknowledging children's mathematics learning in the home and the community and of working with families to understand and build on this, is emphasised as a key aspect of pedagogy. Anthony and Walshaw argue that mathematical proficiency is appropriated by children as they engage in the range of learning interactions described above. The main features of good mathematics pedagogy arising from both Anthony and Walshaw's report and that of the NRC, discussed below, are displayed in Table 1.1.

Principles that Emphasise Learning

Three principles which were derived from a synthesis of work on learning are considered from the perspective of mathematics education (NRC, 2005). The principles are as follows:

- teachers must engage children's preconceptions
- understanding of mathematics requires factual knowledge and conceptual frameworks
- a metacognitive approach enables children to monitor their own learning and development.

While these principles focus on individual learners, we saw above that those presented by Anthony and Walshaw focus more on the context of learning. As such, the two sets can be seen as complementary. Fuson et al. (2005) argue that the strands of mathematical proficiency map directly to the NRC principles:

> *Principle 2 argues for a foundation of factual knowledge (procedural fluency) tied to a conceptual framework (conceptual understanding), and organised in a way to facilitate retrieval and problem solving (strategic competence). Metacognition and adaptive reasoning both describe the phenomenon of ongoing sense making, reflection, and explanation to oneself and others…the preconceptions students bring to the study of mathematics affect more than their understanding and problem solving; those preconceptions also play a major role in whether students have a productive disposition towards mathematics, as do, of course, their experiences in learning mathematics.* (p. 218)

As teachers work with these principles, children's mathematical proficiency is supported. Below we discuss the pedagogical features implied by each of these three principles.

Engaging Children's Preconceptions

The first NRC principle relates to the belief that all new understandings are constructed on a foundation of existing understandings and experiences. One implication of this is that educators must familiarise themselves with children's early mathematics experiences and understandings. We know that there is great diversity in these experiences and understandings (e.g., Tudge & Doucet, 2004), and this can present considerable challenges for teachers, especially at key transitional points such as entry to preschool or school. This suggests a pedagogy which enables educators to find out about children's experiences (individual and collective, as appropriate) and to consider how these influence subsequent learning. A second implication is that teachers also need to ascertain, on a regular basis, children's current levels of understanding as well as their individual ways of thinking, in order to plan appropriately. Encouraging math talk (talking about mathematical thinking) is important in this regard (Fuson et al., 2005). Another implication, and in some circumstances the only option, is close observations of children's engagement in a range of learning activities, and reflection on these from a mathematical point of view (Ginsburg, 2009b). Björklund and Pramling Samuelsson (2012) identify challenges in working with the youngest children (aged 3), as those associated with taking account of children's perspectives on particular concepts and then directing their attention to more developed ways of understanding the same concept. She argues that good mathematics pedagogy is such that materials and learning objectives are specified in advance, that children have opportunities to play and explore and they are challenged with appropriate questions. As they become engaged with the materials and the activity, the educator uses the opportunity to direct children's attention and interest towards learning objectives while at the same time displaying sensitivity and responsiveness to individual children's ways of engaging with the situation.

The idea of intentionally guiding children towards effective strategies is one that needs to be emphasised for educators since it is a relatively new view of the role of educators working with the youngest children. As described in the NRC report (2009, p. 226), 'intentional teaching' is the skill of 'adapting teaching to the content, type of learning experience, and individual child with a clear learning target as a goal'. These emphases were not deemed important for 3- and 4-year-old children until relatively recently. It is likely that emphases which take the lead from the child may present a challenge to some primary teachers also, since many of these may be more familiar with a direct instruction approach.

Integrating Factual Knowledge and Conceptual Frameworks

The second NRC principle relates to the essential role of the elements of factual knowledge and conceptual frameworks in understanding. Both factual knowledge and conceptual frameworks (organising concepts) are important and are inextricably linked. Together they promote understanding, and this, in turn, affects the ability to apply what is learned. For example, a child, from learning the count sequence, may know that four comes after three in the sequence of numbers (at this stage 'four' is still a relatively abstract and shallow concept), but through the repeated use of 'four' in diverse mathematical practices, the concept deepens and connects with other related ideas. In this instance, children's learning can be promoted when the educator intentionally guides children towards considering the range of ways in which we use the number word 'four', i.e. not just in counting but in quantifying, labelling and ordering. The educator will exploit opportunities that present themselves but will also structure activities so that curriculum goals can be promoted. Helping children to link new learning to something they already know enables them to make connections. The process of making connections is very important for young children (e.g., Perry & Dockett, 2008) and contributes towards the development of mathematical proficiency (NRC, 2001) (see Chapter 3, Section: *Connecting*). Counting the sides of a square helps children to connect number to geometry (NAEYC/NCTM, 2002/2010). Thus, good pedagogy emphasises both factual and conceptual knowledge.

Good pedagogy is one that engages and challenges children. It draws on learning paths to help children make progress towards curricular goals (see Report No. 17, Chapter 5, Section: *The Development of Children's Mathematical Thinking*). Knowing about learning paths, and the critical ideas along these, helps educators to support children's progress along the paths. It involves constructing 'bridging' activities and developing conceptual supports to help children make links between math words, written notations, quantities, operations and so on. Teachers need to understand possible preconceptions that children may hold, and they also need to be aware of possible points of difficulty. The importance of the teacher addressing these issues in a proactive way is strongly emphasised in the literature (see Report No. 17, Chapter 5, Section: *Curriculum Development and the Role of Learning Trajectories*). Good pedagogy emphasises the necessity of guiding children (the group and individuals) through the learning paths, ensuring a balance between learner-centred and knowledge-centred needs (Fuson et al., 2005). Ginsburg (2009b) describes how, over a number of years, children supported in this way advance beyond their informal, intuitive mathematics to develop the formal concepts, procedures and symbolism of conventional mathematics. Examples of these paths and their use in assessment are discussed in greater detail later in this report (see Chapter 2, Section: *Formative Assessment*) and also in Report No. 17 (see Chapter 5, Section: *Curriculum Development and the Role of Learning Trajectories*; Chapter 6, Section: *Supporting Children's Progression with Formative Assessment*).

Promoting a Metacognitive Approach

The third NRC principle relates to the importance of self-monitoring, or self-regulation. Self-regulation is supported by children's ability to internally monitor and strategically control actions, as they attempt to undertake a task or solve a problem. According to Vygotsky (1978), self-regulation is promoted through interactions with more experienced others who model and articulate their successful strategies. For example, in learning to complete a jigsaw a child may first witness others (adults or peers) use strategies such as turning pieces, trying pieces, focusing on the shape, size or colour of pieces. Speech accompanying these actions may then be internalised by the child to provide self-monitoring or self-regulating strategies that can later be called on to solve similar problems. An important self-regulatory ability is that of gradually becoming able to talk oneself through similar tasks using external speech but soon moving to internal speech or abstraction (e.g., Berk & Winsler, 1995). Such an approach can help young learners take control over their own learning. One of the ways that they do this is by setting goals for themselves and by checking their own progress towards those goals.

The development of metacognitive awareness i.e. the awareness and control of one's own learning and thinking, helps children to become self-regulated learners. The recognition that children aged 3–6 years can engage in metacognitive processes is relatively recent (see Coltman, 2006 for a review of relevant literature). Organising the classroom environment and the learning activities in particular ways, with an emphasis on particular styles of discourse and interactions between adults and children and between the children themselves is critical. These factors can make a significant contribution to helping children to become independent and self-regulated learners (Whitebread, 2007). Asking children to explain their thinking contributes to the development of metacognition. As children learn to self-monitor they develop and use a meta-language to describe and express their thinking, i.e., a language that includes phrases such as 'I knew they were going to fall down' or 'I counted to see how many'. In one study in the UK (Coltman, 2006), video recordings provided undisputable evidence that children used a wide range of mathematical meta-language as they engaged with planned play activities designed to encourage mathematical talk. This was often a surprise to the educators who worked with them. Examples included 'I am going to fill all this page with numbers' (metacognitive knowledge), 'There are too many hexagons' (strategic control) and 'This is fun isn't it' (motivation). The children also showed, through their talk, an awareness of themselves and others as learners. For example, they made statements about what they were or were not able to do, or demonstrated skills such as counting to other children involved in the play. Good pedagogy, in this instance planned play activities, facilitates self-regulated mathematical learning through verbal interactions which encourage and support a focus on strategic awareness and metacognitive thinking.

Encouraging self-assessment is an important aspect of supporting self-regulation by young children, since it focuses children on thinking about cognitive processes and helps them, for example, to identify errors and monitor thinking (see Coltman, 2006 for a review). Supporting children's self-assessment can be done through using appropriate questions. For example, in the context of constructing a particular structure using blocks, the questioning might include probes such as, 'What made you decide to make your bridge using those particular blocks? Is there any other way that you might build it?'. In relation to selecting the most efficient strategy for sharing out a purse of coins so that everyone has an equal amount, probes might include questions such as, 'Are you sure that everyone has the same amount? How do you know?' (see also Report No. 17, Chapter 6, Section: *Conversations*; Report No. 18, Chapter 3, Section: *Reasoning*). Recognising mistakes, self-correcting, checking, and justifying decisions are some of the behaviours that educators can encourage and develop in order to support children towards realising their capabilities in respect of self-monitoring. These behaviours are closely related to the development of adaptive reasoning, a strand of mathematical proficiency.

The examples above indicate how the learning environment can be structured to support self-regulation. But educators can also foster a metacognitive approach by supporting children's engagement with processes such as estimation and by recognising the role that number sense plays for children learning to check on the feasibility of their responses to number-based problems (e.g., Fuson et al., 2005).

Features of Good Mathematics Pedagogy

Both the NRC report (2005) and Anthony and Walshaw's research synthesis (2007) emphasise the importance of frameworks or systems for thinking about teaching, learning and the design of learning environments. Pedagogy then is seen as a complex whole, a set of connected parts. Isolated practices are not the focus; rather, it is the way in which the different elements of the system interact that is important. Both reports emphasise the learning environment, but the latter also explicitly foregrounds people and relationships, while the former explicitly focuses on the learner.

In Table 1.1 below we list the main features of good mathematics pedagogy as drawn from a combination of both approaches. We group them under the headings of *People and Relationships*, *The Learning Environment* and *The Learner.*

Table 1.1: Features of Good Mathematics Pedagogy

People and Relationships
Strong interpersonal relationships within the setting are fundamental to children's progress.
The classroom atmosphere is one in which all children are comfortable with making contributions.
The diverse cultures of children and their families are taken seriously and treated as classroom resources.
Co-construction of mathematical knowledge is developed through the respectful discussion and exchange of ideas.
The Learning Environment
The starting point for teaching is children's current knowledge and interests.
Classroom activity and discourse focus explicitly on mathematical ideas and problems.
Tasks are designed based on children's current interests, but they also serve the long-term learning goals.
Children are given opportunities to engage in justification, argumentation and generalisation. In this way, they learn to use the language of mathematics.
A wide range of children's everyday activities, play and interests are used to engage, challenge and extend their mathematical knowledge and skills.
Learning environments that are rich in the use of a wide range of tools that support all children's mathematical learning.
Children are provided with opportunities to learn in a wide range of imaginative and real-world contexts, some of which integrate and connect mathematics with other activities and other activities with mathematics.
Investigative-type activities that stem from children's interests and questions, give rise to the creation of models of the problem which can be generalised and used in other situations.
Contexts that are rich in perceptual and social experiences are used to support the development of problem-solving and creative skills.
Children experience opportunities to learn in teacher-initiated group contexts, and also from freely-chosen but potentially instructive play activities.
The potential of everyday activities such as cooking, playing with mathematical shapes and telling the time is recognised and harnessed.
Opportunities are balanced for children to learn in small groups, in the whole-class group and individually, as appropriate.
Teaching is based on appropriate sequencing. Whilst learning paths are used to provide a general overview of the learning continua of the group of children, this is tempered with the knowledge that children do not all progress along a common developmental path.
Planned and spontaneous learning opportunities are used to promote mathematics learning.

Table 1.1: Features of Good Mathematics Pedagogy (*continued*)

The Learner
Children's reasoning is at the centre of instructional decision-making and planning.
Teaching is continually adjusted according to children's learning and as a result of on-going assessment.
Scaffolding that extends children's mathematical thinking is provided while children's contributions are simultaneously valued.
Opportunities are provided for children to engage in metacognitive-like activities such as planning and reflecting. In doing so, children are supported to set their own goals and assess their own achievements.
Assessment is carried out in the context of adult-child interactions and involves some element of sustained, shared thinking.

Ginsburg and Golbeck (2004) expressed their concern that the teaching of mathematics should be responsive to the variations in settings in which early childhood education takes place. This variation is present in terms of the profile of children attending these settings: in age, in cultural background, in language, and in ability. It is also a feature of the profile of educators working in these settings. There are also structural variations between settings, for example in group sizes. We argue that good pedagogy, the features of which are outlined above, can encourage responsiveness by diverse educators working in diverse settings to children. At the same time, identifying features provides a means of promoting good pedagogy across the age range. Anthony and Walshaw (2009a) raise the issue of continuity since they noted a disconnection between available research literature concerning preschool mathematics, and the internationally recognised mathematics education journals and conference proceedings. The authors concluded that all of this suggested a need for some bridging conversations and partnerships between and across education settings, researchers and teachers. These conversations are necessary to achieve the continuity of approaches across settings seen as centrally important for children's mathematics learning, especially at critical points such as starting school (e.g., Perry & Dockett, 2008). In Ireland, children aged 6–7 years are expected to make another transition at the point where they move from senior infants to first class and issues pertaining to this later transition are also important. Conversations between educators might focus on how the features of good pedagogy are realised in everyday activities in the different early education settings, and with children of different ages.

Conclusion

In recent research, a number of important principles (Anthony & Walshaw, 2007; NRC, 2005) which underpin the features of good mathematics pedagogy for young children have been identified. One set of principles focuses on people and relationships, and the learning environment. A complementary set of principles focuses on learning and includes the engagement of children's preconceptions, the integration of factual knowledge and conceptual frameworks, and the promotion of a metacognitive approach. All of these principles are consistent with the aim of developing mathematical proficiency. Through combining these, we identify a comprehensive account of the features of good mathematics pedagogy. The broad sets of principles are illustrated in Fig. 1.1 below:

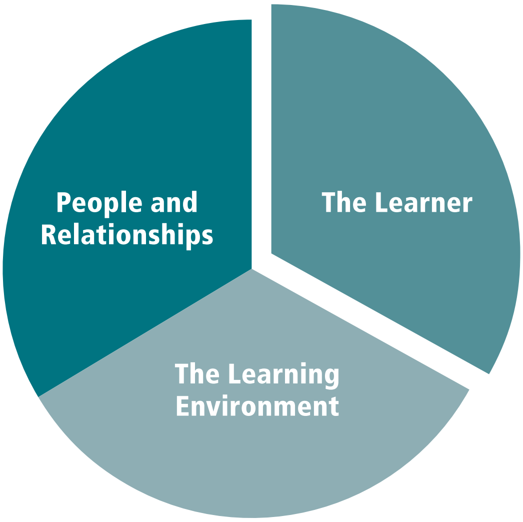

Fig. 1.1: The Connected Elements of Good Mathematics Pedagogy

However, it is impossible to think about good mathematics pedagogy for children aged 3–8 years without acknowledging that much early mathematical learning occurs within the play environment. It is also crucial to identify how adults help children maximise mathematics learning through play. Learning through play is seen as fundamental to good mathematics pedagogy in early childhood. It assumes varying degrees of emphasis depending on the age of the child. There is a gradual transition to more formal approaches as children move through primary school.

The key messages arising from this chapter are as follows:

- Educators need to understand how mathematics learning is promoted by young children's engagement in play, and how best they can support that learning.
- The features of good mathematics pedagogy can be identified with reference to robust principles related to people and relationships, the learning environment and the learner.
- The principles and features of good mathematics pedagogy for children aged 3–8 years pertain to all early educational settings, and are important in promoting continuity in pedagogical approaches across settings.

CHAPTER 2

Teaching Practices

Guidelines for educators (e.g., NAEYC/NCTM, 2002/2010; AAMT/ECA, 2006) recommend that, for early mathematics education to be effective, teachers need to use a variety of practices and materials to support children's mathematical learning. The role adopted by the teacher is viewed as crucial. The teacher enables the learning to take place by structuring the environment and involving children in a variety of learning experiences (e.g., Pound, 1999). Successful educators build on children's interests and experiences by engaging in a wide range of teaching practices to support children's mathematical understanding. Practices highlighted later in the section (play, story/picture-book reading, project work, learning through the arts, drama and physical education, and the use and integration of tools including digital technologies) are ones that exemplify how good pedagogy is enacted in the course of everyday activities in early education settings. Each of these practices reflects a number of the features of good pedagogy as identified in Table 1.1. In addition to highlighting features of good pedagogy for educators, we have identified five overarching meta-practices that are essential in promoting mathematical thinking and understanding, and that are important in helping children towards achieving the overall aim of mathematical proficiency. These meta-practices should permeate all learning activities if optimal mathematical learning and development are to be promoted.

Meta-Practices

The five meta-practices discussed below are: promotion of math talk, development of a productive disposition, mathematical modeling, use of cognitively challenging tasks, and formative assessment. There are many others we are sure, but looking back at Report No. 17, each of these five meta-practices emerged as important in relation to pedagogy.

Promotion of Math Talk

The centrality of sustained interactions for deepening and extending children's understandings in all aspects of their learning is an issue that has received a great deal of attention in recent years, mainly due to research such as that carried out as part of the *Researching Effective Pedagogy in the Early Years* (REPEY) (Siraj-Blatchford et al., 2002). This research points to a need, in early childhood, for extended discussion with individual or small groups of children. Such opportunities create the conditions for sustained shared cognitive engagement between educator and child and for ensuring optimal cognitive challenge for all children (e.g., Anthony & Walshaw, 2009a). Skilful and thoughtful questioning of children is also a feature of pedagogy highlighted in the REPEY report, and by a number of early years' mathematics experts (e.g., Ginsburg, 1997; Copple, 2004). In Report No. 17 the issue of questioning was addressed in the context of formative assessment (see Chapter 6, Section: *Methods*). Skilful questioning and sensitive interventions are pedagogical strategies that have important roles to play in moving children from 'I don't know why' responses, to responses where they focus on critical aspects of the problem under consideration (e.g., Casey, 2009).

Children talking about their mathematical thinking and engaging in mathematization are identified as important ways for them to make their thinking visible (Fuson et al., 2005). It is particularly significant in supporting the growth of young children's conventional mathematical knowledge over time. Consequently its development is regarded as a key focus of early mathematics education (e.g., Ginsburg, 2009a; Perry & Dockett, 2008). However, we know that the amount of teachers' math-related talk varies, with qualitative differences in that provided by teachers in different classrooms (Klibanoff et al., 2006). As described in Report No. 17 (see Chapter 3, Section: *The Nature and Scope of Mathematical Discourse*), this aspect of pedagogy involves an explicit focus on language which conveys mathematical ideas related to, for example, quantity, shape, size and location. It also involves encouraging and supporting children's communication, and their initial efforts to engage in reasoning and argumentation.

The teacher has a key role to play in providing a model of the language that is appropriate in a particular mathematical context. Recasting everyday experiences using mathematical words and phrases is a key element of inducting children into talking about their mathematical thinking. Children need to be assisted in using the newly-acquired mathematical language in their descriptions, explanations and justifications. Good mathematics pedagogy recognises that some children (e.g., children living in disadvantaged circumstances; children who speak a language that is different from the language of instruction) may experience difficulty with problems presented in verbal format and there may be a need to adjust the presentation accordingly (e.g., Ginsburg, Cannon, Eisenband, & Pappas, 2006). It also takes account of the general path of development and tailors expectations of the form and extent of children's responses accordingly. Specific strategies include using children's own stories in teaching mathematics; integrating language that is familiar to children in teaching mathematics; promoting children's first language; encouraging think-aloud strategies; and integrating non-linguistic materials to facilitate maths language (Lee, Lee, & Amoro-Jimenez, 2011).

The challenge of eliciting talk about mathematics with the youngest children should not be underestimated. Many young children respond intuitively to mathematics problems but they may need support in articulating their reasoning or in justifying a solution in the conventional way (e.g., Dunphy, 2006) (see Chapter 3, Sections: *Justifying*; *Reasoning*). Digital tools as 'objects to think with' can lead to situations wherein children externalise their thinking (see section on *Digital Tools* below).

Work in establishing math-talk learning communities in classroom settings (e.g., Hufferd-Ackles, Fuson, & Sherin, 2004) provides a blueprint for strengthening the focus on language as a tool for teaching and learning mathematics, and, in particular, for developing children's understanding of concepts, strategies and mathematical representations. Such communities support children's efforts to develop understanding, to engage in mathematical reasoning, and to communicate their mathematical ideas. Math-talk learning communities have a strong social dimension, with children sharing thoughts with others, and listening to others sharing ideas (Chapin et al., 2009). Research in the Irish primary context has documented how students build on each other's mathematical ideas in lessons and across lessons and how a communal sense of responsibility for learning is developed (NicMhuirí, 2012).

Chapin et al. (2009) outline key teaching practices associated with improving the quality of mathematical discourse. These include:

- Using 'talk moves' or strategies that engage children in discourse, including revoicing (where the teacher clarifies his/her understanding of the child's contribution), asking a child to restate someone else's reasoning, asking a child to apply their own reasoning to someone else's ideas, prompting for further participation, and using wait time effectively (see Dooley (2011) for examples in an Irish context).
- Using effective questioning to support key mathematical goals such as engaging children in reasoning mathematically (e.g., 'Does this always work?') and making connections between mathematical ideas and their application (e.g., 'What ways have we used to solve this problem?'). It is also important to help children to rely on themselves to determine whether something is mathematically correct (e.g., 'How do we know?').
- Using children's thinking to propel discussion, including identifying children's misconceptions, enabling them to figure out those misconceptions themselves, being strategic about who shares during discussion, and choosing ideas, strategies and representations in a purposeful way that enhances the quality of the discussion.
- Setting up a supportive environment that enhances children's engagement in mathematical discourse – e.g., by providing relevant visual aids, mathematical tools and mathematically-related vocabulary.

- Orchestrating the discourse, through such practices as anticipating children's responses to challenging mathematical tasks, monitoring their work on and engagement with tasks, selecting particular children to present their mathematical work, and connecting responses to key mathematical ideas (see also, Smith et al., 2009).

The work of Hufferd-Ackles et al. (2004) highlights the complexity of transitioning from a traditional approach to mathematics teaching, in which the teacher takes centre stage, to a discourse community, in which children make key contributions to developing their own mathematical understanding as well as that of their classmates. To support teachers in making this transition, Hufferd-Ackles et al. have produced developmental trajectories[3] that address four aspects of mathematical discourse: questioning, explaining mathematical thinking, sources of mathematical ideas, and responsibility for learning. The trajectories show intermediary levels along which math-talk communities develop, and allow teachers to address difficulties and dilemmas as they move along. The levels can be summarised as follows:

- **Level 0** – teacher-directed classroom with brief answer responses from children.
- **Level 1** – teacher begins to pursue student mathematical thinking. Teacher plays central role in math-talk community.
- **Level 2** – teacher models and helps children build new roles. Some co-teaching and co-learning begins as child-to-child talk increases.
- **Level 3** – teacher functions as co-teacher and co-learner…teacher monitors all that occurs, still fully engaged. Teacher is ready to assist but now in a more peripheral and monitoring role (Hufferd-Ackles et al., 2004, pp. 88–90).

The literature points to significant challenges that teachers can encounter in implementing math-talk learning communities in their classrooms. One of these is regression to earlier levels on the trajectory when a new topic is presented as teachers may need to occupy a more central role in introducing new concepts, vocabulary or procedures (Hufferd-Ackles et al., 2004). NicMhuirí (2012), who attempted to facilitate this type of classroom community in an Irish primary classroom, also noted a tension between making student thinking an object of classroom discourse and maintaining coherent lessons and sequences of lessons (see also Fernandez, Yoshida, & Stigler, 1992). Teachers' knowledge of pedagogy for teaching mathematics is also important (see Chapter 6, Section: *A View of Mathematics*), since, without such knowledge, teachers may not be able to identify children's misconceptions, identify opportunities for extending their thinking and moving them along a learning path, or support them in making connections between existing knowledge and new mathematical ideas, or across aspects of mathematics (Anthony & Walshaw, 2007).

3 Unlike the trajectories/learning paths presented elsewhere in this volume, these trajectories describe development from the perspective of both teachers and children.

Dooley (2011) noted that teachers may need to reconceptualise their sense of efficacy (defined as their sense of their ability to take effective action in teaching) as they make the transition from approaches to teaching that emphasise 'telling' or 'initiative-response-evaluation' patterns, to dialogic approaches that emphasise math talk. Drawing on the work of Smith (1996), she noted that teacher efficacy could be developed, not only through generating and directing discourse (math talk), but also through selecting appropriate and relevant problems, predicting children's reasoning, and judicious or selective telling. Importantly, Dooley also noted that patterns of discourse in which teachers encourage children to explain their thinking, and focus their attention on what is not yet understood reveal greater equity in the teacher-child relationship, compared with approaches that are mainly characterised by telling or evaluation. While her work involved older primary-school children, the general principles also apply when working with younger children.

Math talk can be nurtured in a range of learning contexts including whole-group settings, small groups (e.g., collaborative learning groups) and pairs. On some occasions, teachers may assign specific tasks to groups to work on together (for example, classifying a set of shapes, solving a problem together). On other occasions, children may be asked to discuss a problem or work on the wording of an explanation in pairs, for a limited period of time. These practices allow for an increase in children's engagement in math talk, and provide the teacher with opportunities to monitor one or more groups or pairs, and gather and use information about their learning.

Development of a Productive Disposition

In Report No. 17 (see Chapter 1, Section: *A Key Aim of Mathematics Education: Mathematical Proficiency*) we saw that individuals who have a productive disposition believe that mathematics is useful and relevant and an area of learning in which they can engage successfully. Disposition has been identified as an important aspect of learning in the domain of mathematics which is acquired over time (De Corte, 2007). In early childhood, productive disposition begins with the fostering of a positive disposition towards the mathematics that they encounter in their everyday life. Bertram and Pascal (2002, p. 94) describe dispositions in early childhood as '…environmentally sensitive'. They are acquired from and affected by interactive experiences with the environment, significant adults and peers…positive dispositions are learnt but they are rarely acquired didactically'.

This implies a focus on experiences initiated by children and developed by educators (see the discussion on play and on project work below). Children play an active role in the development of their dispositions by participating and collaborating in mathematically-rich activities. Children's eagerness to participate in everyday activities such as cooking (Vandermaas-Peeler, Boomgarden, Finn, & Pittard, 2012), or shopping is an effective way of fostering positive disposition, especially in circumstances where the adult is sensitive to children's interests and preferences. As a result of her study of the number sense of 4-year-old children, Dunphy (2006) concluded that the ways in which children are engaged with mathematics, how they view mathematics and the contexts in which mathematics are presented to them are what shape their dispositions towards mathematics. In the same study, children with a positive disposition also demonstrated a strong number sense.

A productive disposition can be fostered by educators who draw children's attention to the various aspects of mathematics, and who engage children in what to them are interesting and relevant experiences that show the usefulness of mathematics for solving everyday problems. For instance, young children starting school may already have developed a liking or enthusiasm for number, based on experiences during the preschool period, i.e., their disposition towards number is already developing (e.g., Dunphy, 2006). It is essential therefore, that children's experiences with mathematics in early education settings are ones that are engaging and challenging (e.g., see the discussion on story/picture-book reading and the mathematically-related discussion arising from this, later in the chapter). This message has important implications for the pedagogical practices used by educators. The practices used should enable children's agency and incorporate their interests and preferences.

It is important to stress that in these early years disposition is still quite malleable, and the early experiences at preschool and school are likely to be critical for some children. Hence, curriculum guidelines should emphasise that issues related to disposition (e.g., the learning environment, opportunity to participate) need to be investigated by teachers, and systematically supported so that all children can develop a productive mathematical disposition.

Emphasis on Mathematical Modeling

The idea of a mathematical model as it is generally used in mathematics education stems from the way it is used in the discipline of mathematics, that is, as a quantitative or spatial system that can be used in particular, prescribed ways. From this perspective, the model is seen as existing independently of individual or collective activity. Base-ten blocks (i.e., Dienes' blocks) that are used in the teaching of number operations are an example of a mathematical model in this context. The teacher is the expert who has knowledge of the mathematics represented by the model and the intention is to use the model to make the mathematics accessible to the children.

The idea of a model is used in a different way within the Realistic Mathematics Education (RME) approach. Here models emerge as individuals interact with particular activities (see Report No. 17, Chapter 5, Section: *The First Approach: Working with Children's Thinking and Understanding (RME)*). Gravemeijer and Stephan (2002, p. 148) say that (in the RME approach) 'modeling is seen as an organising activity from which the model emerges' and that 'subsequent acting with these models will help the students (re)invent the more formal mathematics that is aimed for'. An example of this is where the empty number-line can be used to model children's informal strategies for addition. A child might intuitively respond to a problem involving the sum of 26 and 18 by using a strategy such as 26 + 20 - 2. For many children such intuitive methods are coherent with their emerging number sense. Base-ten blocks, the conventional materials used in many classrooms for multi-digit addition, do not easily lend themselves to modeling these intuitive strategies. The number-line, as described here, supports children's strategies and encourages the development of increasingly sophisticated ones. In this case the model is being used to fit with, rather than to steer children's thinking. While at first children use the empty number-line as a model of their informal solution strategy (a model *of* a situation), gradually they become able to use the number-line for thinking about mathematical relations between numbers (a model *for* thinking about number relations). Proponents of the RME approach argue that working in this way, children develop deeper and more flexible understandings that can be applied to a range of situations.

English and her colleagues put a somewhat different emphasis on models and modeling (English, 2007; English & Sriraman, 2010). From their perspective, models are 'systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other familiar system' (English, 2007, p. 121). For them, modeling problems 'are realistically complex situations where the problem solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artefacts or conceptual tools that are needed for some purpose, or to accomplish some goal' (English & Sriraman, 2010, p. 173). Thus, from this perspective, mathematical modeling is an approach in which problem-solving is not separate from but integral to the understanding and development of new concepts. English and Sriraman (2010) regard mathematical modeling as an advance on the usual problem-solving that occurs in schools because:

- it often involves quantities or operations that go beyond those encountered in word problems
- it encourages children to mathematize as they try to make sense of a particular situation
- it uses contexts that draw on several disciplines
- it encourages the development of a model (e.g., graph or table) that can be applied to a range of situations
- it encourages social interaction and collaboration as students usually work in small groups or teams to solve the problems.

While much of their research on mathematical modeling in primary school concerns older children, English and Sriraman maintain that it has a rightful place in the very early years where important foundations for future learning of mathematics are laid. In particular, they propose the development of statistical reasoning through mathematical modeling and suggest as an example the pursuit of a question such as, 'Is our own playground fun and safe?'. In order to develop models to address this question, children engage in an iterative fashion in (i) refining questions and identifying attributes; (ii) measuring attributes and recording initial data; (iii) organising, analysing, interpreting, and representing their data; and (iv) developing data-based explanations, arguments, and inferences, and sharing these with their peers. Other questions might arise out of this work such as, 'How can we make our playground safer?'. It seems to us that this perspective on modeling shares many emphases with those of the *Project Approach* which is discussed later in this chapter.

Thus, while their focus is on modeling as addressing realistically complex situations rather than modeling as an organising activity of more usual mathematics problems (see Gravemeijer & Stephan above), they align with the RME perspective in their emphasis on mathematization and on the development by the learner of a model that can be used in a variety of situations. We see that both have a significant role to play in a redeveloped mathematics curriculum for 3- to 8-year-olds. We suggest that this can best be conveyed in the curriculum presentation by the provision of detailed exemplars illustrating the two interpretations of modeling.

Cognitively Challenging Tasks

Stein, Grover, and Henningsen (1996) define a mathematical task as an activity 'the purpose of which is to focus students' attention on a particular mathematical idea' (p. 460). They maintain that the tasks used in classrooms are integral to the kinds of mathematical thinking in which students engage, and therefore to learning outcomes. They make particular reference to 'cognitively challenging tasks' as a means to promote higher-order thinking. Sullivan, Clarke, & Clarke (2013) argue that the overarching aim of mathematical proficiency implies using a variety of task types – an implication of this is that emphasis is not placed exclusively on worked examples that predominate textbook activities (and that are aligned with the development of procedural fluency). Account is also taken of rich and challenging activities that build on what children know mathematically and experientially, that allow them time and opportunities to make decisions, and that foster collaboration and communication. Drawing on a large body of research, Anthony and Walshaw (2007) draw the following conclusions about tasks:

- Open-ended tasks support student thinking and exploration. The openness relates to a range of 'correct' solutions and/or a range of ways to achieve one or more solutions.
- Tasks should provide students with opportunities for success, present an appropriate level of challenge and promote student agency and personal interest.
- When designing and implementing tasks, it is important that the goals and activities are responsive both to individual students' levels of understanding and to the discipline of mathematics.
- Differentiation can be facilitated by providing the same basic task to all students and taking individual needs into account (e.g., extra supports, extension activities etc.).
- Productive task engagement requires that tasks are closely linked to a student's current level of knowledge and understanding but are 'just beyond' his or her cognitive reach.

- In order to make tasks accessible, it is important that they are set in contexts that are 'realistic', that is, that allow learners to think in 'real' ways. The contexts can be real or imaginary settings that illustrate how mathematics is used. Some studies have found that the use of contexts can disadvantage children (particularly those in low-SES communities) who may be more literal in their interpretation of the problem situation (Cooper & Dunne, 2000). This does not mean abandoning realistic contexts but rather avoiding tasks that use mathematics to solve problems in unrealistic ways or those that use unrealistic or unfamiliar situations.
- Tasks can remain cognitively challenging throughout a lesson if emphasis is placed on ways of thinking rather than on correct procedures, if sufficient time is allocated to completion of the task and if there is a continued emphasis by the teacher on justification and explanation.

Formative Assessment

The NAEYC/NCTM (2002/10) position statement on early childhood mathematics states that, in providing high-quality mathematics education for young children, teachers and other professionals should 'support children's learning by thoughtfully and continually assessing all children's mathematical knowledge, skills, and strategies' (p. 9). The statement emphasises the importance of assessment when planning for ethnically, culturally, and linguistically diverse young children and for children with special needs. It also emphasises the use of assessment outcomes to plan and adapt teaching and curriculum. It notes that young children may invent their own mathematical ideas and strategies, which are quite different from those of adults, and that these need to be recognised. The NRC (2009) report, *Mathematics Learning in Early Childhood*, links the use of formative assessment (observation, tasks, interviews) to intentional or planned teaching, with assessment outcomes informing decisions about future learning.

In Report No. 17 (*Chapter 6*), we reviewed a range of formative assessment methods that can provide valuable information about young children's mathematical development, though it was stressed that it might be appropriate to use multiple methods on some occasions (e.g., an observation or task followed by an interview). The methods, which are consistent with the approach to assessment described in *Aistear* (NCCA, 2009a), include:

- *Observations* – structured observation of a child's engagement in mathematics. Learning stories (Carr, 2001) were identified as an approach to recording observations that could include a child's dispositions.
- *Tasks* – pre-designed or teacher-designed activities that provide insights into a child's mathematical understanding (e.g., Yelland & Kilderry, 2010).
- *Interviews* – focused conversations that explore in depth children's thinking and reasoning through questioning (and observation), generally about tasks that the child undertakes as part of the interview (e.g., NRC, 2009).

- *Conversations* – frequent but less-detailed questioning about a child's mathematical thinking, arising in the course of completing tasks or other activities.
- *Pedagogical documentation* – dialogue and reflection by teacher and child on a range of artefacts (e.g., pictures, recordings, work samples) that arise from engagement in mathematical tasks (MacDonald, 2007).

The research literature notes that relatively few studies have demonstrated clear links between assessment outcomes, planned instruction, and growth in children's mathematical learning (e.g., NRC, 2009), but that learning paths are a framework that teachers can draw on for the assessment of children's learning (see Report No. 17, Chapter 6, Sections: *Interviews*; *Conversations*; *Supporting Children's Progression with Formative Assessment*).

Formative assessment was highlighted in Report No. 17 as being most consistent with sociocultural, child-centred approaches to mathematics education, and the unsuitability of more summative assessment measures for use with young children was noted. On occasion, formative assessment information can be complemented by information derived from screening or diagnostic tests.

Practices in Integrative Contexts

Good pedagogy is enacted in the course of everyday activities in early education settings, and it is characterised by the features of good mathematics pedagogy identified in Chapter 1 (see *Table 1.1*). Pedagogical practices discussed in this section focus on engaging children in play, in story/picture-book reading, in project work, and on mathematics learning through arts or physical education. These provide some important contexts in which young children in early educational settings engage with mathematical ideas. Other contexts, for example, problem-solving in specific content areas of mathematics, are considered in Chapter 3. Moreover, there are opportunities for mathematics development across all areas of learning, and not just the ones discussed in this chapter. For example, spatial concepts and spatial relations can be developed through exploring a geographically-focused theme.

The practices highlighted here promote children's use of a range of tools, including digital tools. The learning activities arise from children's interests, concerns, and questions and the educator links these to learning goals. The practices are generally holistic in nature and facilitate an integrated approach to mathematics education for children aged 3–8 years. However, a clear focus on mathematical goals is required, even within an integrated approach. As emphasised earlier in the introduction of this chapter, it is essential that the meta-practices discussed above (the promotion of math talk, development of a positive disposition, emphasis on mathematical modeling, use of cognitively challenging tasks, formative assessment) permeate all learning activities if children are to develop mathematical proficiency. The relationships among these elements are illustrated in Figure 2.1.

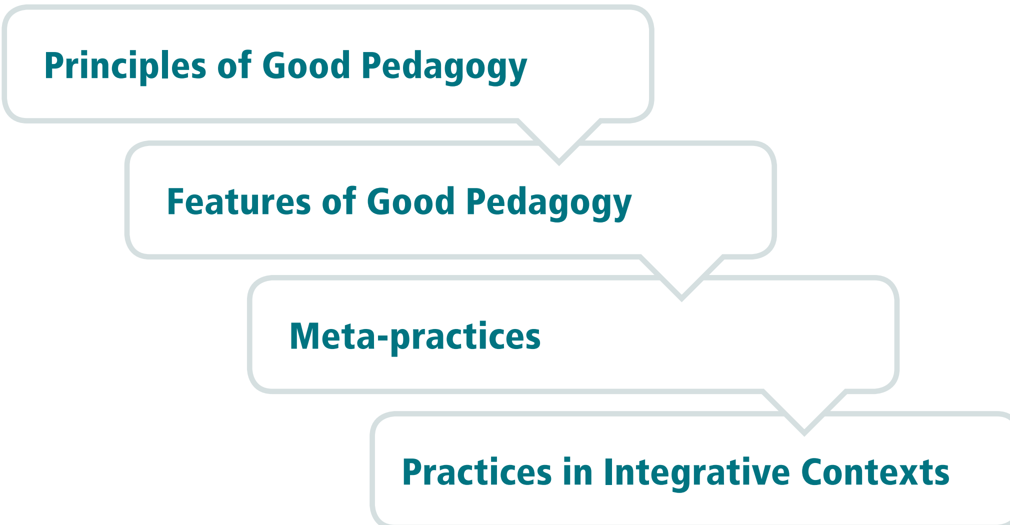

Figure 2.1: Relationships between Different Aspects of Pedagogy

Play

Given the importance of play as a learning process for young children, it is essential that good mathematics pedagogy recognises this fact, honours it and harnesses its power. Sarama and Clements (2009) identify three types of play in which children engage with mathematics: sensorimotor play, symbolic or pretend play, and games with rules. *Aistear* (NCCA, 2009a) promotes a range of different types of play, i.e., 'creative', 'games with rules', 'language', 'physical' and 'pretend'. Although not outlined specifically in *Aistear*, all of the above types of play contribute in their own way to children's mathematical learning and can offer valuable opportunities for playful mathematical experiences (Ginsburg et al., 2006; Perry & Dockett, 2008).

The various types of play strengthen children's mathematical learning and understanding in different ways. The following examples highlight ways in which mathematical skills and concepts can be developed in early years settings, in both indoor and outdoor environments:

- Physical play refers to physical, exploratory, manipulative and constructive play. It is the most common type of play in very young children (Montague-Smith & Price, 2012) as it involves bodily movements such as clapping, hopping and jumping. Through engaging in physical play experiences, children can learn a variety of mathematical concepts and skills. Physical play experiences include participating in games and activities that develop the vocabulary of position and movement; identifying and comparing shapes and patterns within the environment;

exploring and manipulating materials and identifying their characteristics; and comparing sizes of objects and counting them. Through engaging in constructive play children develop mathematical skills such as problem-solving, visualisation, spatial awareness and reasoning, tessellation and pattern-making.

- Pretend play encompasses make-believe, dramatic, socio-dramatic, role, fantasy and small world play. Pretend play involves children being creative and using their imaginations with objects, actions and in role-playing. Through participating in pretend play, children develop early literacy and numeracy skills. Through playing with real objects they develop mathematical skills and engage with concepts such as number operations related to counting, calculating, problem-solving, number, measure and time. Using objects to symbolise other things, children move from thinking in the concrete to thinking in the abstract (NCCA, 2009a).
- Creative play involves children exploring actions and materials and communicating their ideas. Through creative play children develop a variety of mathematical skills in meaningful contexts. For instance, children playing with junk and recycled materials can make models, explore the properties and characteristics of 2-D and 3-D shapes, investigate symmetry and tessellation and develop mathematical reasoning and problem-solving by constructing and deconstructing shapes.
- Language play involves children playing with sounds and words. Children learn mathematical language through discussion in playful situations, e.g., shopping, cooking or number stories. When children engage in play they can use objects to symbolise or create something new and, in doing so, can use mathematical language associated with the new object. Through counting concrete materials in playful contexts number language can be extended.
- Games with rules include activities where children follow a specific set of instructions or negotiate their own rules. Games with rules provide opportunities for collaborative learning and for the development of mathematical activities including reasoning, problem-solving, classifying and ordering. These activities can include people games with children following directions such as ‘Simon Says’, games measuring time such as ‘What time is it Mr Wolf’, movement games and number and board games. For example, in ‘Simon Says’ children might be asked to clap three times, or take two steps then one step, altogether three steps. Accommodations should be made for language levels. In invented games children can select appropriate manipulatives to support their learning e.g., dice, playing cards and number cards.

The playful activities above contribute to the development of aspects of mathematical proficiency such as conceptual understanding and productive disposition. They also present valuable opportunities for observation and assessment of mathematical understanding and learning.

These play activities listed above are, for the most part, teacher-initiated and directed. When planning for mathematical development through playful activities, educators need to be also mindful of the fact that child-led play offers rich opportunities for mathematical learning and understanding. *Aistear* (NCCA, 2009a, p. 53) stresses that children love to make choices about when, what, where, how and with whom to play. Educators should ensure that quality resources are available so that as they play, young children can construct and reinforce mathematical knowledge. Through engaging with these quality resources children can, for example, construct a model, identify numbers in the play environment, exchange coins for goods, find a block to fill a space and choose blocks to copy a sequence or a pattern.

Despite the strengths of play as outlined above, it is also recognised that not all playful activities lead to mathematical understanding (Ginsburg, 2006). Research indicates that children do not always engage in mathematical learning opportunities as play can often be restricted by such contextual factors as lack of resources, curriculum overload, limited space, and class or group size (Kernan, 2007; McGrath, 2010). Children's dispositions, arising from their experiences, might also be implicated here (e.g., Dunphy, 2006), and some children may need encouragement and support to engage in a mathematical way in play, or in mathematical play. Another limiting factor may be the undervaluing of the potential of play by adults, who may be under pressure to provide evidence of specific types of learning (Wood & Attfield, 2005). These limitations also present pedagogical challenges for the educator who attempts to implement a play-based mathematics curriculum.

Despite the challenges outlined above, it is evident that play is a key process through which children learn mathematics and practitioners can overcome many of the challenges in ensuring that optimum outcomes for children by careful and resourceful planning. Observing children at play, thinking creatively about play spaces and resources both indoor and outdoor, participating and interacting in playful situations, co-constructing with children and assessing the effectiveness of play experiences are all aspects of pedagogy which are essential to productive and worthwhile mathematical play-based experiences for children (e.g., Kernan, 2007).

Story/Picture-Book Reading

Picture-Books

Research indicates clearly that children's literature contributes greatly to the process by which young children acquire mathematical thinking. It does so by offering enjoyable and meaningful contexts – paper-based or digital – in which mathematical content and concepts may be explored and developed (Casey, Kersh, & Young, 2004; Hong, 1999; van den Heuvel-Panhuizen, 2012). Literature for young children generally includes pictures since artwork is an important feature in the education of pre-literate children. In most story books the illustrations, as well as the text, play a prominent role in the telling of the narrative and the creation of meaning (Elia, van den Heuvel-Panhuizen, & Georgiou, 2010) so these books are generally referred to as 'picture-books' (van den Heuvel-Panhuizen & Elia, 2012). Picture-books usually show mathematical concepts visually (Murphy, 1999) and therefore support children's understanding of abstract concepts (Montague-Smith & Price, 2012). Through engagement with picture-books, young children are presented with rich contexts in which they encounter problematic situations, ask questions, reason mathematically and have conversations with adults and peers, all of which can lead to the use of mathematics-related language (Anderson, Anderson, & Shapiro, 2005; Hong, 1996; van den Heuvel-Panhuizen, 2012; Young, 2001).

A number of studies have examined how the use of picture-books enhances young children's mathematical understanding. Hong (1996) investigated the impact of a programme that focused on mathematics-related storybook reading, discussion, follow-up activities and play on children's performance in specific mathematically-rich tasks. Her findings revealed that the 4- to 6-year-old children involved in the study did significantly better on tasks involving classification, number combinations and shape compared to the control group. It was suggested also that the experimental group were more favourably disposed towards mathematical learning and thinking, and chose to spend more time engaging in mathematical tasks and in the mathematics corner.

Research by Young-Loveridge (2004) indicated that 5-year-old children involved in an intervention programme which focused on listening to number stories and rhymes, as well as playing number games scaffolded by adults, demonstrated significant improvements in numeracy skills when compared with a group who were not involved in the programme. An important aspect of this programme was the fun the children had engaging in activities. This illustrates the role that parents have in supporting and enhancing the mathematical abilities of children, especially during the transition to school.

Interactions during Story/Picture-Book Reading

Casey et al. (2004) endeavoured to embed mathematics in a story context through the use of six problem-solving adventure stories. The texts were designed to develop the children's spatial and analytical skills. Their results indicate that the children who encountered geometry within a storytelling context using one of the above books achieved greater success in their mathematical activities with blocks than those who did not engage with similar content within a story context. In a follow-up study using one of these picture-books, Casey, Andrews, Schindler, Kersh et al. (2008) investigated the use of block-building interventions to develop children's spatial-reasoning skills. A puppet was used as the story-teller and his presence provided a meaningful context for the carrying out of mathematically-related tasks. Another study showed that girls benefitted more from geometry interventions than boys (Casey, Erkut, Cedar, & Young, 2008). This is an important finding given that we know that girls, regardless of age, are at a disadvantage in solving spatial problems compared with boys (e.g., Casey, 2009). However, it may be the quality and depth of the spatial language environment experienced by girls, rather than exposure to specific spatial activities, that are more critical for girls' early acquisition of spatial skills (Dearing et al., 2012). It therefore appears that story-book reading combined with play, activity and focused language development provide for optimal learning.

Children can be mathematically engaged by listening to a picture-book being read aloud, even without additional teacher intervention (van den Heuvel-Panhuizen & van den Boogaard, 2008). In this study an analysis of children's spontaneous mathematically-related speech during a story-reading session – using a book not designed specifically to teach mathematics – indicated that they used both spatial orientation-related utterances and number-related utterances. In a study by Elia et al. (2010), a picture-book that was written specifically for the purpose of teaching mathematics was used. As in the previous study, the teacher did not give explicit instruction or question the children as she read the story. The findings again revealed that the children used mathematics-related speech. This was thought to be due to the fact that they were presented with a context that made sense to them.

However, this is not to underestimate the effect of appropriate teacher intervention. For example, Elia et al. also reported that pictures with a representational content were found to elicit mathematical thinking to a greater extent than pictures that included informational functions. Therefore, while pictures in stories may be perceived as valuable tools in enhancing children's mathematical thinking, there needs to be adult interaction if children are to benefit fully from the mathematical-informational purposes of a story. Björklund and Pramling-Samuelson (2012) stress that even though mathematical concepts may be embedded within a story context, many children cannot recognise these. They suggest that the three important factors that must be borne in mind when storyreading are: shared attention, reasoning and meaning, and the teaching of specific mathematical content (all of which have the potential to contribute to the development of mathematical proficiency). In the case of the latter, they argue that the teacher must have an intended mathematical objective when reading a story if the children are to achieve optimum learning from the story context. Indeed, van den Heuvel-Panhuizen and Elia (2012) stress that the reading style that best suits the power of the picture-book to develop children's mathematical thinking and understanding is dialogic book reading (e.g., Cunningham & Zibulsky, 2011). Here the emphasis is on letting the picture-book provide the context for the co-construction of meaning between child and adult, with the balance of power in favour of the child.

Pramling & Pramling-Samuelsson (2008) carried out a study using a storytelling context where children were required to solve a mathematical problem and to represent their answers through illustrations. A key finding of the study is the need to make explicit to children different ways of representing mathematical information. The authors caution against the inclusion of extra resources by teachers, which in this case were pictures, in an effort to support young children's problem-solving. Rather than supporting the children's learning, the pictures created uncertainty as the children were attracted to the incidental rather than the critical in the story.

Selecting Books

As the above studies indicate, picture-books vary in the amount and types of mathematical knowledge they present. Thatcher (2001) identifies criteria for selecting books for teaching mathematics, and offers advice on the effective use of literature in the teaching of mathematics to young children. In a more recent study, van den Heuvel-Panhuizen and Elia (2012) draw on extensive research literature to examine basic issues in relation to the characteristics of picture-books that support young children's mathematical understanding. They used recent findings and theory to produce a framework of learning-supportive characteristics of picture-books for learning mathematics. Their framework, presented below, should prove useful to those who wish to evaluate the suitability of certain picture-books for young children's mathematical development (van den Heuvel-Panhuizen & Elia, 2012, p. 34).

Learning supportive characteristics of picturebooks for learning mathematics

I. Supply of mathematical content

I.1. Mathematical processes and dispositions

The picturebook shows mathematical processes
- Solving problems with mathematical knowledge
- Using mathematical language and representations
- Reflecting on mathematical activities and results
- Mathematical reasoning

The picturebook shows mathematical dispositions
- Eagerness to learn and enquiring attitude
- Tenacity in solving problems
- Sensitivity to the beauty of mathematics

I.2. Mathematical content domains

The picturebook deals with

I.2.a. Numbers-and-counting
- Counting sequence
- Ordering numbers
- Determining numerosity of collection (resultative counting), estimating, ordering/comparing numbers, representing numbers, operating with numbers (adding, subtracting, multiplying, dividing)
- Contextualising numbers (giving meaning to numbers in daily life situations), positioning numbers (indicating where a number is on a numberline) or structuring numbers (decomposing or factorising)

I.2.b. Measurement
- Different ways of measuring: directly measuring, pacing out units of measurement (natural units or standardised units), using measuring tools, representing and interpreting measuring results, using reference measures
- Dealing with different physical quantities such as length, volume, weight, time

I.2.c. Geometry
- Orienting: localising, taking a particular point of view, rotations and directions
- Constructing: concretely constructing objects and visualising constructions (explaining how a building is built, reproducing a building), properties of spatial and plane shapes
- Operating with shapes and figures: geometrical transformations (shifting, mirroring, rotating, projecting, and combinations of these)

I.3. Mathematics-related themes

The picturebook deals with
- Growth
- Perspective
- Fairness
- Ratio
- Order (in time, of events)
- Cause and effect
- Routes

II. Presentation of mathematical content

II.1. Way of presenting

The mathematical content ...
- is addressed explicitly (something mathematical is happening that is explained) or is addressed implicitly (something mathematical is happening that is not explained)
- is integrated in the story (either explicitly or implicitly) or is isolated from the story (e.g., there is a picture of somebody wearing a dress with a nice geometrical pattern, but the story does not mention this dress)

II.2. Quality of presentation

II.2.a. Relevance

The picturebook ...
- contains mathematical content that is valuable for children to learn
- offers mathematical content that is presented in a meaningful context (the contexts make sense, are worthwhile, contain natural connections with other subjects)
- shows mathematics that is correct (misconceptions should be avoided; however incorrect things and inaccuracies can be learning-supportive under particular conditions)

II.2.b. Degree of connection

The picturebook ...
- connects mathematics with children's life and world
- connects mathematics with interests of children
- makes connections between mathematics and reality
- shows the coherence between mathematical concepts and connects different appearances and representations of mathematics
- establishes relationships between mathematics and other subjects

II.2.c. Scope

The picturebook ...
- makes understanding possible at different levels
- offers multiple layers of meaning
- anticipates future concept development

II.2.d. Participation opportunities

The picturebook ...
- offers opportunities to make children actively involved in the picturebook (prompts children to do something by themselves)
- draws in children passively (makes them listen and observe)
- stimulates particular modalities (engages the children cognitively, emotionally, or/and physically)

by means of ...
- Asking questions: questioning or posing problems, asking open-ended questions, presenting challenges, conflicts, changes of perspectives, ambiguities, or mistakes
- Giving explanations: explaining mathematical content, giving hints or clues, visualisations, describing experiments, including repetition or accumulations
- Causing surprise: showing astonishment, tension, including jokes, surprising events, provocative language, offering a reward

Figure 2.2: Framework of learning-supportive characteristics of picture-books for learning mathematics

Experiences with mathematically-related stories have the potential to promote aspects of mathematical proficiency, including procedural fluency, adaptive reasoning and a productive disposition.

Project Work

The *Project Approach* is recognised as offering opportunities for mathematical development. The term 'project' refers to an in-depth study of a particular topic undertaken by small groups of children (Katz & Chard, 2000). It is designed to assist young children to make deeper and fuller sense of events and experiences and to support their learning by encouraging them to make decisions and choices in collaboration with their peers and teachers (Katz, 1998). Children's interests provide the stimulus for the topic or project to be investigated. The *Project Approach* presents children with opportunities to make sense of real-life problems (NAEYC/NCTM, 2002/2010) as most projects involve a wide variety of types of problem-solving (Helm & Katz, 2010). Children's mathematical concepts and language may be developed across subject matter boundaries (NAEYC/NCTM, 2002/2010).

Projects involve children in investigating a topic of interest or importance to them. The impetus for the project comes from the children themselves. A key feature of a project is that it is an investigation that encourages the active participation of children in the planning, development and assessment of their own work (Katz & Chard, 2000). The essence of the *Project Approach* is to engage children in a complex and interesting project that exploits and elaborates on the mathematics that arise in the course of the activity (Ginsburg & Golbeck, 2004).

The roots of the *Project Approach* can be traced to the work of Dewey and Kilpatrick (Hall et al., 2010). In Reggio Emilia schools, the word *project* has a broader meaning in that it involves investigation and expression by means of various symbolic languages (Tarini, 1997). Project work has been described and advocated in the *Primary School Curriculum* (Government of Ireland, 1999a) and *Aistear* (NCCA, 2009a). The Primary School Curriculum emphasises that for young children the distinctions between subjects are not relevant. It stresses the importance of a coherent learning process where connections are made between learning in different subjects. The theme of *Exploring and Thinking* in *Aistear* (NCCA, 2009a) focuses on children making sense of their environment. *Aistear* highlights the importance of the role of the adult in project work. The adult enhances the children's learning experiences during the project process by providing resources, participating in project-related activities and interacting with children. The adult showcases the children's projects (by displaying photos or showing video) and helps them share their work with other children and parents.

The Project Approach in Action

Boaler (1997) asserts that the *Project Approach* enhances children's problem-solving skills as they are consistently challenged to solve mathematical problems that occur as the project unfolds. According to Helm and Katz (2010), the *Project Approach* helps children to

- generate an awareness of the function of number and quantity concepts
- create a reason to quantify information
- represent quantities with numerals
- see reasons to classify and sort
- develop categories
- use tools for investigation, experimentation and observation
- compare and order objects
- engage in mathematical thinking
- use measurement, counting and graphing
- develop an awareness of shape, area, distance and volume
- construct models, drawing diagrams and charts and creating play environments.

The *Project Approach* weaves mathematics with young children's everyday experiences in the early education setting and offers rich opportunities for the development of mathematical thinking and understanding. Of particular relevance is the incorporation of digital tools in young children's projects. As discussed in Report No. 17, using technology is an increasingly important avenue of learning and expression for children. For example, Kalas (2010) describes children's engagement with technology and digital tools as they pursued a project on *My Town*. As the project developed, children explored direction and location using techno-toys on their specially-constructed floor map. In doing so, they explored spatial concepts and developed the language of spatial relations (e.g. beside, towards). As they investigated long/short and longest/shortest routes, there were opportunities to develop algorithmic thinking (processes or rules for calculating). Children also documented their peers using a video-camera, and this was used as a basis of discussion, explaining, reasoning and justification. This project can be seen to provide rich opportunities to develop a number of aspects of mathematical proficiency. Further examples of rich mathematical activities or investigations drawn from the literature are outlined below.

A study of water (adapted from Dixon, 2001)

Children form groups that focus on different aspects of water e.g., 'What can water do?'; 'Where does water come from?'. Children decide on suitable activities and experiments and conduct these within their groups, for example, experiments relating to capacity and sinking and floating. Children document processes through diagrams, drawings, charts, photographs, data and models. Children demonstrate activities to the rest of the group and, in doing so, explain mathematical processes.

Making apple sauce (adapted from Ginsburg & Golbeck, 2004)

Children decide how many jars of apple sauce are required; they count the number of jars; they 'read' a pictorial recipe for apple sauce; they discuss ingredients to purchase; they walk to the supermarket and discuss the route; they weigh ingredients; they compare size, shape, colour and price of fruits; they exchange money for apples and calculate change and, on returning to the school, they make the apple sauce which leads to further investigations.

The pizza project (adapted from Gallick & Lee, 2009)

Children discuss the topic of food, recipes for pizza and the development of a 'topic web' based on these. They also sequence the making of a pizza; estimate, measure and cut circles of paper to represent pizza slices; develop a pizza-themed play area; order and pay for pizza; and share pizza amongst friends. From a teaching and learning perspective, projects are a valuable approach to organising mathematical activities for young children (Katz & Chard, 2000; Ginsburg & Golbeck, 2004). While some learning experiences may look like projects, a learning experience cannot be considered to be a project unless the elements of child initiation, child decision-making and child enjoyment are present (Helm & Katz, 2010). It can be seen that project work, carefully implemented, can develop each of the strands of mathematical proficiency.

Learning Mathematics through the Arts and Physical Education

This section examines specific ways in which links can be established between the arts (music, the visual arts, drama) and mathematics and sets out options for how the mathematics curriculum for 3- to 8-year-olds might seek to strengthen links between mathematics and other areas of learning (and vice versa).

Music

Music is a rich context in which educators can develop children's mathematical language and concepts. Shilling (2002) suggests that, through a classification of sounds and movement, children's mathematical understanding and skills are enhanced. She identifies a strong link between the order, timing, beat and rhythm of music and attributes of mathematics such as counting, sequencing and understanding time and order. The integration of music into children's mathematical and physical activities supports their logical and rhythmic development and enables teachers to make learning both music and mathematics more meaningful for the children (Kim, 1999; McGrath, 2010; Montague-Smith & Price, 2012; Pound, 1999; Shilling, 2002). Engaging children in making and responding to music may also contribute to the development of other skills and attitudes that are important for mathematics such as concentration, creativity, perseverance, self-confidence, and sensitivity towards others (Fox & Surtees, 2010).

Among the ways in which music and mathematics are linked are the following:

- Young children come to school with intuitive knowledge of musical patterns and rhythms (Shilling, 2002). Their first musical experiences can often include lullabies, nursery rhymes, stories and songs. Teachers can create mathematical opportunities for children to respond to the rhythms, patterns and sequences embedded in music.
- Children can learn and practice counting through the recitation of rhymes, chants and songs that have counting-related words (Kim, 1999).
- The development of number sequences is achieved through repeated rhymes, songs and stories. Children move on to associate a value to these number names (McGrath, 2010; Montague-Smith & Price, 2012; Pound, 1999).
- In making and responding to music, children should have an opportunity to create a range of musical patterns and to understand such musical elements as pitch (gradations of high/low), dynamics (gradations of volume, louder/quieter, silence), tempo (different speeds) and structure (the ways different sounds are organised) (Fox & Surtees, 2010). Fox and Surtees show how teachers can specify particular mathematics objectives as they plan lessons or projects involving music. They also suggest that teachers consider how aspects of mathematics can be assessed in cross-curricular contexts such as music.

Visual Arts

Pattern and shape are key features of both the visual arts and mathematics. In the visual arts, children encounter colour, form, texture, pattern and rhythm, and shape (Government of Ireland, 1999c). In mathematics, they discover patterns of number and shape, symmetry, tessellation, and the properties of a range of 2-D and 3-D shapes.

A key aim of the visual arts curriculum is 'to develop the child's awareness of, sensitivity to and enjoyment of visual, aural, tactile and spatial environments' (Government of Ireland, 1999c, p. 4), while awareness of the visual and spatial qualities in the environment is also important for mathematical understanding, and for enhancing children's ability to apply mathematical knowledge in the environment (i.e., in real life). Although the current visual arts curriculum provides specific suggestions for linkages with other areas of the curriculum, just a few of these relate specifically to mathematics. The following are some ways in which mathematics might be integrated into the visual arts:

- identifying 2-D shapes (circles, triangles, rectangles, squares) in fabrics
- repeating patterns, translation and rotation
- measuring fabric samples and investigating perimeters
- looking and responding: identifying and talking about geometric patterns and symmetry in pictures

- identifying light and dark areas
- using ICT to design and discuss the properties of a print.

With the youngest children, play with malleable materials such as clay or dough is rich in opportunities to experience the way in which a given quantity can change shape. The educator can use the opportunities presented to help children to understand the meaning of, for example, measure words such as *long* and *short*. Children can be supported to use and refine key vocabulary, as appropriate, to include words such as *longer than* (comparatives) and *longest* (superlatives). Activities such as printing allow children to begin to develop concepts of area and perimeter, and in this context they may also make connections with patterning as they experiment with sequencing elements and/or groups of elements and repeating sequences to form patterns.

Drawings and mark making can be used by children to convey their growing awareness of number and quantity. The educator, in considering the child's verbal explanations of the graphics he/she creates, can gain insight into the child's current and developing understandings of the ways in which we use mathematical language and record this by making marks. Through discussion, the child also develops abilities to translate mathematics from one language (verbal) to another (graphic) (Worthington & Carruthers, 2003).

Again, Fox and Surtees (2010) argue that integration is most effective when specific learning outcomes are identified and consideration is given to how achievement of the objectives can be assessed. They also point out that the arts provide a forum in which children's confusions about particular aspects of mathematics can be addressed (e.g., the base of 2- and 3-D shapes does not need to be parallel to the bottom of the page – orientation can vary; length is not necessarily longer than width).

Drama and PE

Role-play offers many opportunities for children to engage with mathematical concepts and skills. Story contexts such as 'The Three Little Pigs' can give rise to a range of mathematically-related play, especially if appropriate props are provided to stimulate mathematical thinking (e.g. Pound, 2008). The educator can develop concepts through discussion as appropriate. For example, in many role-play contexts children can be challenged to consider questions about quantity. Phrases such as 'just enough' (equality), 'not enough' (less than) and 'too many' (greater than) can be used and their meaning explored in the context of the play.

Although the current curricula in drama and physical education do not emphasise specific approaches to integrating these areas with mathematics, they give rise to authentic contexts that can be used to develop children's understanding of mathematics, for example:

- Games which involve throwing beanbags into a hoop, bouncing a large ball, skipping and then counting to answer the question 'How many?'.

- Forming groups for games, representing basic processes such as addition or subtraction, by combining or separating groups of children. Partitioning of numbers can be explored – for example, a group of 7 children could explore the different ways in which 7 could be partitioned by splitting into two subgroups (6 + 1; 5 + 2 etc.).
- Creating 2-D shapes such as triangles or rectangles using children's bodies, and discussing the properties of such shapes.
- Visualising the properties of 3-D shapes such as cylinders, cuboids and triangular prisms by pretending to reside inside a shape and describing the sides, angles and corners and showing how to travel inside the different shapes.
- Examining and discussing the movement involved in dance to identify lines, shapes, pattern and symmetry.
- Participating in swimming or athletics and calculating times and distances. Very young children can be exposed to mathematical vocabulary through everyday discourses such as swimming lessons (Jorgensen & Grootenboer, 2011 as reported in MacDonald, Davies, Dockett, & Perry, 2012).
- Engaging in problem-solving activity in role-play.

Clearly, much can be gained from linking aspects of the arts and PE curriculum with mathematics. Key issues for curriculum development include the following:

- Should curricula in the arts and PE specify in more detail the connections with mathematics that can be made, by, for example, identifying learning outcomes in mathematics that can be achieved through activities in the arts?
- Should the mathematics curriculum include learning outcomes that relate specifically to application of mathematics in other curriculum areas?
- Should teachers be expected to assess children's ability to integrate mathematics into other subject areas? Should this be done as part of assessing mathematics? Or separately?

Digital Tools

In Report No. 17, we gave some attention to the role of tools in the construction of mathematical knowledge. We discussed how, from a sociocultural perspective, tools – including both physical artefacts and symbolic resources – are an integral aspect of human cognition and activity. Cultural tools are considered to influence the ways in which people interact with and think about the world (see Report No. 17, Chapter 2, Section: *Sociocultural Perspectives*). The physical artefacts include manipulative materials, pens, books and computers, while symbolic resources include language, drawings and diagrams (Armstrong et al., 2005). Elsewhere in this report we give consideration to tools such as language (for example, the section on *Promotion of Math Talk* in this chapter) and concrete materials and drawings (for example, the section on *Representing*, Chapter 3).

Digital tools deserve particular attention because of their key role in children's lives. However, schools and teachers are generally not systematically incorporating curriculum or NCCA guidelines on digitally-related activity in mathematics into the work that takes place every day in classrooms (DES, 2010, p.12). Furthermore, children in junior classes experience a narrower range of digitally-related activity than children in senior classes (DES, 2008). Beyond Ireland, many researchers (Perry & Dockett, 2004; Perry, Lowrie, Logan, McDonnell, & Greenlees, 2012) acknowledge that there is a dearth of evidence-based research investigating young children's use of technology in mathematics learning. The available research tends to focus on screen-based technologies, calculator use or the role of the teacher (Fox, 2007; Mulligan & Vergnaud, 2006; Perry & Dockett, 2007b; Yelland, 2005). However, some researchers have begun to examine the potential of computer-based tools for mathematical representation by young children (Clements & Sarama, 2007; Highfield & Mulligan, 2007; Moyer, Niezgoda, & Stanley, 2005).

Digital Technologies as Learning Tools

'Children born…[today] are growing up in a world in which digital technologies are not only widely accessible to most families living in Western societies, but so commonplace as to be unremarkable' (Plowman, Stephen, & McPake, 2010, p. 135). Our classroom environments need to reflect this ubiquitous presence so that young children can play with and experience these digital tools that have cultural significance in order to gain a sense of empowerment and control over the technology (Price, 2009). Research has demonstrated that the use of these tools has the potential to significantly improve educational opportunities for young children (Price, 2009; Siraj-Blatchford & Whitebread, 2003) and can benefit young children's learning in a range of ways (Downes, 2002; Clements & Sarama, 2003; Haugland & Wright, 1997; Plowman & Stephen, 2005; Yelland, 2005, 2007; Zevenbergen & Logan, 2008).

If these tools are used as 'an object to think with' (Papert, 1980) or a 'mindtool' (Jonassen & Carr, 2000), young children can develop higher-order thinking and engage in knowledge construction. These tools enable children to revisit and reflect on their prior learning, so that they can become more actively engaged in the learning processes (Bauer & Kenton, 2005). This reflection 'allows further learning to be sequentially linked and re-constructed in the light of previous thinking' (Highfield, 2010a, p. 181). Digital tools thus have the potential to assist in the development of children's mathematical proficiency, particularly in relation to the strands of adaptive reasoning, strategic competence, and productive disposition. However, a major challenge facing early childhood educators is to begin thinking about digital technologies as learning tools which children learn 'with' and not 'from' (Jonassen, Howland, Moore, & Marra, 2003).

Research indicates that young children's technology play has been one of the most contentious issues faced by early childhood education in recent decades. At the centre of this debate have been ideas of developmental appropriateness and fears that technology use creates risks for social and emotional development (Cordes & Miller, 2000; Healy, 1998; Highfield, 2010a). Some fear that communication might be inhibited by technology play but this has been challenged by many researchers (e.g., Kelly & Schorger, 2001; Hyun & Davis, 2005). Indeed, Clements and Sarama (2004) take the opposite position, stating that 'computers are catalysts of social interaction' (p. 341).

Many of these fears stem from a restricted view of technology (e.g., a focus on desktop computers), which in turn can lead to a restricted view of play. Yelland (2010) calls for a re-conceptualisation of play to incorporate activities using new media as playful experiences that are supported by adults. She argues that exploration in virtual worlds requires us to rethink the nature of play. This contemporary view of play incorporates new technologies that afford opportunities for young children to play and communicate in multiple modes so they are able to acquire deeper understandings about how things work and connect and are relevant to their lives. Arising from the literature, we offer some representative examples of the use of digital technologies for supporting early mathematical development. Since much of the research is based on specific tools and software, these feature in the examples below.

Example 1: Techno-Toys
Technology has enabled the creation of a new generation of techno-toys that differ from traditional toys as they have embedded electronics, response systems and microchips that enable them to respond to children in some way. They can be categorised by their technical features or by their 'affordances' (functions and engagements that a toy may enable) which can be intended or unintended, as well as 'open-ended' (allows users to engage in child-controlled creative processes, e.g., *BeeBots*) or 'closed' (only allows user to respond in limited ways). Highfield (2010a) developed a classification system which incorporates potential possibilities afforded by techno-toys. These include opportunities for young children to: represent and create; manipulate; program; communicate; investigate; simulate and model; problem-solve and think strategically; and play a rules-based game. Highfield (2010b) outlines five scenarios of children's digital play with techno-toys. The activities in which the children engaged included patterning, number and numeric structure, spatial awareness and positional language, size, ratio and proportion, and time. In one of the scenarios, Highfield describes the use of BeeBots, a techno-toy that has potential to support the mathematical development of children aged 3–8 years.

BeeBots are simple robotic toys whose movements can be programmed by children. This affordance of programming the BeeBot resonates with the research on Logo which indicated Logo's usefulness as a tool in teaching and learning mathematics (Butler & Close 1989, 1990; Clements & Sarama, 1997; Hoyles, 1987; Hoyles & Noss, 1992; Yelland, 1995), particularly in relation to the development of geometry and spatial concepts (Clements & Battista, 1992). By interacting and playing with the BeeBots, children are developing mathematical ideas (e.g., spatial awareness, positional language, ideas of directionality, concepts of measurement, estimation, counting, and transformational actions, including linear movement and rotation) and metacognitive processes (planning, problem-solving and reflection) (Highfield & Mulligan, 2008; Highfield, 2010b).

It has also been suggested that techno-toys such as the BeeBots have the potential to advance progress along learning paths by exposing children to concepts that they normally would not be introduced to until a later age. However, it must be realised that exposure to advanced ideas does not ensure that children will grasp and understand these ideas or concepts as they may just 'wash over' the children without application or understanding (Highfield, 2010a).

Example 2: Software

Virtual manipulatives (e.g., pattern bocks, base-ten blocks, geo-boards, tangrams etc.) are 'an interactive, web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge' (Moyer et al., 2002, p. 373). They provide access to unlimited quantities of materials and can be used to help children develop ideas of composition and decomposition of number as well as patterns and relationships (Reimer & Moyer, 2005; Moyer et al., 2005). Virtual or digital manipulatives can draw on children's intuitions about physical objects and extend those objects to allow a new range of concepts, which were previously viewed as too advanced to be explored (Resnick et al., 1998). The use of virtual manipulatives has also been examined by Swan and Marshall (2010), who confirm the findings of an older study (Perry & Howard, 1997), concluding that 'their use did not guarantee success: the major benefit of the manipulatives comes from the discussion that goes on around them and explicit linking by the teacher to the mathematics they represent' (Perry et al., 2012, p. 177).

Teachers need to be able to capitalise on the sometimes unintended affordances of software. One example of this is a drawing package called Kidpix (*http// www.broderbund.com*), which is dynamic interactive software. With this, simple patterning can be collaboratively explored, as the tool's primary affordance is in creating and representing, while the secondary affordance is manoeuvring and manipulating images around the screen. Highfield (2010a) argues that traditional drawing techniques do not support such dynamic interaction and thus Kidpix may offer new learning opportunities for children. The software provides opportunities for geometric actions such as flips, rotations, shearing and scaling (Highfield & Mulligan, 2008, p. 19). As with all computer software, children can save their work, adding to it and changing it as they wish, so that future learning can be informed easily by prior experiences. Teachers can also use freely accessible tools to enable children to express their understanding of particular concepts. For example, using AutoCollage, children can easily construct a montage from the images they capture using the digital camera to illustrate shapes they see in the environment, e.g., sets of different numeric value, patterns observed in nature, insects with a set number of legs, etc.

Example 3: Interactive White Board

The interactive white board (IWB) has become a popular tool in Irish primary classrooms over the last five years in particular. However, in many settings, IWBs tend to be used by teachers predominantly as a replacement for the traditional blackboard rather than capitalising on its interactive possibilities for children's learning. There appears very little research on its efficacy in the context of early learning of mathematics. While a small number of studies look at the use of the interactive white board for supporting young children' s mathematical development (e.g., Goodwin, 2008), much work needs to be done in this area. Given that the software which supports the use of the interactive white board often includes a bank of virtual manipulatives, observations made above about the use of virtual manipulatives should also be borne in mind when using the interactive white board in classroom settings.

In summary, there needs to be a concentrated effort for further research to move beyond just screen-based tools and examine how the full range of existing and emerging digital tools and computational devices can make powerful mathematical ideas accessible to and impact on young children's mathematical and meta-cognitive processes. However, this research also needs to be mindful of the learning environment and the complex role of the teacher for, as Clements (2002) points out, 'the curriculum in which computer programs are embedded, and the teacher who chooses, uses, and infuses these programs, are essential elements in realising the full potential of technology' (p. 174).

Conclusion

In Report No. 17, math talk, disposition, modeling, tasks and assessment all emerged as important factors in the theoretical discussions about mathematics education. In this chapter we surveyed the literature which offers a range of perspectives, and advice, as to the issues for educators in incorporating these elements into their practices. We saw that good mathematics pedagogy can be enacted when educators engage children in a variety of activities which have the potential to develop mathematical understanding. The activities should arise from children's interests, questions, concerns and everyday experiences. They may be generated across different areas of learning and they may utilise a range of tools, including digital tools. The potential of these activities for developing mathematical proficiency can best be realised when educators focus on children's mathematical sense-making.

The key messages arising from this chapter are as follows:

- Good mathematics pedagogy incorporates a number of meta-practices including the promotion of math talk, the development of a productive disposition, an emphasis on mathematical modeling, the use of cognitively challenging tasks, and formative assessment. A pedagogy incorporating these meta-practices supports the vision of 'mathematics for all'.
- A deep understanding of the features of good mathematics pedagogy should inform the ways in which educators engage children in mathematics across all areas of learning.
- Educators need to maximise the opportunities afforded by a range of tools, including digital tools, to mediate learning.
- Practices which reflect the features of good pedagogy contribute to the development of the strands of mathematical proficiency.

CHAPTER 3

Curriculum Development

Remillard (2005) maintains that the designers of curriculum materials must take account of the teacher-curriculum relationship and the underlying messages that their materials communicate to educators. In this chapter, attention is given to the overarching idea of mathematical proficiency as an aim of mathematics education for 3- to 8-year-old children, the processes that need to be developed in line with this aim and the content domains that need to be included in a redevelopment of the curriculum. While we recognise that the current PSMC (Government of Ireland, 1999a) encapsulates some of these ideas (e.g., inclusion of process skills), we argue for a reformulation of the curriculum which foregrounds mathematical proficiency as the main aim. Mathematization should be a key focus and its associated processes should be clearly indicated. We also propose a rebalancing of the focus on processes compared with content. Towards the end of this chapter, various ways in which learning paths might be used in formulating the mathematics curriculum are explored.

Although broad aims, process skills and content objectives for the teaching of mathematics are provided in the Irish PSMC (Government of Ireland, 1999a) (a teacher's guide with practical teaching examples is also available – Government of Ireland, 1999b), there remains an overly strong focus on the strand of Number and on implementation of procedures and textbook activities in most primary classrooms (see Report No. 17, Introduction, Section: *Performance Context*; this report, *Introduction*). The reasons for this are complex and manifold. However, a matter in need of attention is the lack of synchronisation between the aims, processes and content objectives. For example, problem-solving is highlighted in the introduction to the curriculum:

> *Developing the ability to solve problems is an important factor in the study of mathematics. Problem-solving also provides a context in which concepts and skills can be learned and in which discussion and co-operative working may be practised. Moreover, problem-solving is a major means of developing higher-order thinking skills…*(Government of Ireland, 1999a, p. 8)

However, in the listing of content objectives, the solution and completion of practical problems are usually placed at the end of the sequence of objectives relating to strand units for a given class. The following, for instance, are the content objectives for the strand unit of length in 1st class (Government of Ireland, 1999a, pp. 52–53):

The child should be enabled to

- estimate, compare, measure and record length using non-standard units;
- select and use appropriate non-standard measuring units and instruments;
- estimate, measure and record length using standard unit (the metre);
- solve and complete practical tasks and problems involving length.

This listing is at variance with the idea of problem-solving providing a context within which concepts and skills can be developed. Rather, the impression given is that children first have to learn procedures and then apply these known procedures to practical situations.

In Report No. 17 we discussed mathematical proficiency as an aim of the curriculum and mathematization as a key focus. In this chapter we extend the discussion of curriculum structure by addressing aims and goals. How these elements relate to each other is an important issue in a redeveloped curriculum. The approach we take here is to give some attention to each of the key processes associated with mathematization. We also give a brief account of each of the five content domains with particular reference to key emphases in recent years.

Curriculum Aims

In their description of the landscape of learning mathematics, Fosnot and Dolk (2001) make reference to learners journeying towards a 'horizon'. We conceptualise this horizon – or aim of mathematics education – as mathematical proficiency. A redeveloped curriculum should serve to realise this aim and goals, coherent with this aim, should be identified.

Curriculum Goals

In Report No. 17 (Chapter 4, Section: *Breaking Down the Goals: Critical Transitions within Mathematical Domains*) we proposed that we first need to identify general goals and these then need to be broken down for planning, teaching and assessment purposes. A goal specification with a strong focus on processes is in keeping with a sociocultural perspective on learning. The presentation of goals in the Dutch mathematics curriculum is of interest to the Irish situation. Their starting point is the characterisation of mathematics education and a statement of core goals for the entire primary mathematics curriculum (van den Heuvel-Panhuizen & Wijers, 2005). To support progression to these goals, the Dutch team have developed learning-teaching trajectories for calculation with whole numbers (van den Heuvel-Panhuizen, 2008) and for measurement and geometry (van den Heuvel-Panhuizen & Buys, 2008). Within these trajectories are intermediate attainment targets that serve as a series of reference points against which children can be assessed. Suitable teaching methods at each stage of the learning process are also provided in the learning-teaching trajectories. Furthermore, the targets are to be used in conjunction with the characterisation of mathematics education and core goals. The Dutch team resists an excessive precision with regard to age and grade-level. This is to avoid frequent testing of children to see if they are meeting targets. Instead trajectories are described for two successive school years in recognition of the fact that children learn at different rates.

As has been outlined in Report No. 17, the differences between the ways learning paths are presented rest largely on their theoretical underpinnings. In the US, Sarama and Clements (2009) draw heavily from the field of cognitive science, whereas the RME team draw from classroom-based research. The developmental progressions described by Sarama and Clements are finely grained and age-related, whereas the TAL[4] trajectories are characterised by fluidity and the role of context (see *Table 5.2. A Developmental Progression for Volume Measurement* and *Table 5.2. A Developmental Progression for Volume Measurement*, Report No. 17). We suggest use of learning paths to explicate critical transitions in relation to the content. We see that in such a formulation, there would be specific reference to processes throughout. In line with a sociocultural approach to the learning of mathematics, we advocate that learning paths be used in a flexible way to posit shifts in mathematical reasoning and to inform planning and assessment.

In the section below we discuss some issues pertaining to process and content-oriented goals for mathematics education. The mathematical processes discussed are those associated with mathematization (a key focus of the curriculum). The content goals discussed are those generally addressed in international curricula.

Mathematical Processes

In Report No. 17, we discussed how mathematical proficiency is developed through engagement with the processes encompassed in the overarching concept of mathematization (Bonotto, 2005; NRC, 2009). The processes – communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting – are described next.

Communicating

Communication is at the heart of mathematics learning. Zevenbergen et al. (2004) describe communication in mathematics from a multi-literacy perspective:

> *In terms of multiliteracies, the mathematics in a classroom is a text of which students will make interpretations (or readings). When teaching is seen in this way, it becomes possible to understand the learner as a much more active participant in the classroom and in so*

4 In Dutch, learning-teaching trajectories are referred to as TALs (i.e., Tussendoelen Annex Leerlijnen).

> *doing allows the teacher to realise that students construct very different interpretations of what has been said or done. This moves the emphasis away from seeing students as giving right or wrong answers to one where the role of teacher becomes more of understanding why students construct responses and understandings in the ways they do. Not only are the communications related to mathematics, but so also are the texts within which the mathematics is being conveyed to the students. Meaning making becomes multi-dimensional.* (p. 117)

Among the communications they identify as relevant to mathematics are:

- oral communication – contexts for this include whole-class discussion, small group work, play, dramatic performances etc.
- visual communication – this might take the form of 2-D displays, constructions, photographs
- digital communication – displays can be created using digital technology, e.g., AutoCollage and Glogster
- textual communication – this includes scribbles, drawings, stories, ways of thinking sheets etc.
- symbolic communication – this involves communicating meaningfully in the symbolic form of mathematics (e.g., +, -, =); as discussed in Report No. 17 (Chapter 2, *Sociocultural Perspectives*) children move from invented to conventional symbol systems.

Reasoning

While there are various accounts of mathematical reasoning (Sternberg, 1999), it is generally associated with logic and the drawing of valid conclusions (e.g., Artzt & Yaloz-Femia, 1999; Steen, 1999). Reid (2002), drawing from the NCTM *Principles and Standards for Teaching Mathematics* (2000), describes three elements that constitute mathematical reasoning in primary school settings: (a) examining patterns and noting of regularities; (b) supporting statements by showing that they apply in other cases or rejecting statements by providing counterexamples; and (c) explaining reasons 'why'. Earlier we discussed the principle of promoting a metacognitive approach as a means of helping children to monitor their own learning and development (See Chapter 1, Section: *Promoting a Metacognitive Approach*). Tang & Ginsburg (1999) suggest that metacognitive ability, i.e., 'thinking about one's thinking', is closely related to reasoning. As such, helping children to understand their thinking and assisting them to express it to others are central to the learning of mathematics. This expression may take many forms. For example, children might use a questioning tone to indicate uncertainty or might smile to convey their belief that they have found a satisfactory solution to a problem. Gestures such as imitating actions, intentionally using gaze, touching and pointing have been identified as key modes of expression for young children (Flewitt, 2005). Educators, therefore, need to pay close attention to issues such as tone of voice, facial expression, gesture, and specific use of words as indicators of children's self-awareness.

Argumentation

Krummheuer (1995, p. 229) describes argumentation as 'a social phenomenon; when cooperating individuals [try] to adjust their intentions and interpretations by verbally presenting the rationale of their actions'. It is considered central to mathematics development because children have to make sense of their own explanations and the explanations of others and have to compare the claims of others against their own (e.g. Perry & Dockett, 1998, 2008; Yackel & Cobb, 1996). Perry and Dockett (1998) suggest that argumentation might occur in a variety of situations; however, play, because of its significance in the lives of young children, offers a particularly potent context in which it might emerge.

Justifying

Related to the idea of justification is that of 'self-explanation' described by Siegler and Lin (2010, p. 85) as 'inferences concerning 'how' and 'why' events happen'. Drawing on a number of mathematics and science experiments with young children, they conclude that preschoolers, as well as older children and adults, can benefit from encouragement to explain their thinking. They also suggest that explaining other people's answers can be more useful for children than explaining their own answers. Not surprisingly, the more time children are given to think about such explanations, the higher will be the quality of their learning. The authors also report that 'I don't know' responses (observed in a study with 5-year-olds) decreased over time (see also Chapter 2, Section: *Promotion of Math Talk*). Moreover, Perry and Lewis (1999) report that verbal imprecision (e.g., false starts or long pauses) can be related to improved problem-solving performance. This imprecision and hesitation in young children's verbal interactions have been found to often indicate their engagement with deep intellectual work (Tizard & Hughes, 1984).

Generalising

Generalisation involves a shift in thinking from specific statements to more general assertions. The fact that children use concrete objects to explore mathematical thinking does not imply that they are not engaged in abstract thought as, in the words of Russell (1999, p. 3), 'the very nature of mathematics is abstract'. She suggests that as children learn to count, they are already dealing with abstract ideas and that this leads to further abstractions (e.g., 'Numbers go on for ever.'). In particular, children often express generalisations using language, diagrams and story contexts (Bastable & Schifter, 2008). For example, a child might say, 'It doesn't matter what way you add two numbers, the answer stays the same' to express the commutative property of addition. In this case 'you' is not used by the child to address another person but to convey generality – what happens 'every time' (Rowland, 2000).

Mason (2008) contends that children begin to generalise from an early age (for example in learning vocabulary such as 'cup', 'dog' etc.) and yet this capacity to generalise is rarely exploited in educational settings:

> *…as teachers, we often try to do the work for them [children]. We provide particular cases, we display methods, and we provide worked examples. We then expect them to generalize, yet rarely do we explicitly and intentionally prompt them to use their powers to generalize, nor display that power being used.* (p. 64)

However, errors in children's mathematical thinking can be caused by the development of prototypes (e.g., only recognising a triangle if it is lying 'flat') or by overgeneralisation (e.g., that a smaller digit must always be subtracted from a larger one), both of which can be countered to some extent by engagement in rich and varied mathematical experiences (Ryan & Williams, 2007). Generalisation is embedded in algebraic thinking which is considered later in this chapter.

Representing

Among the forms of representation that children use to organise and convey their thinking are concrete manipulatives, mental models, symbolic notation, tables, graphs, number lines, stories, and drawings (Langrall et al., 2008). These are sometimes referred to the literature (and earlier in this volume) as 'tools' (see, for example, Anthony & Walshaw, 2007); in other words, the terms are often used interchangeably.

Meira (2002) argues that representations are at the heart of sense-making in mathematics:

> *It is often the case… that mathematics instruction not only restricts students' production of unconventional and specialized notational systems (e.g., when doing arithmetic on paper), but also aims at suppressing the students' 'dependency' on representations altogether (often viewed only as a means to acquire mental competencies). Tallies and diagrams on paper (as well as finger counting and the use of hand calculators) are not lesser means of doing mathematics, but the very material basis of sense-making.* (p. 102)

While representations in their many forms are integral to children's mathematical sense-making, there are some caveats that must be taken into consideration in their use. For example, it has traditionally been considered that there is a linear development from concrete to abstract thinking (Piaget, 1952). The literature on the subject suggests that this is not necessarily the case and that representations or models can sometimes inhibit children's mathematical thinking (for example, Uttal, Scudder, & DeLoache, 1997). This is the case because children do not necessarily understand the relationship between the model (e.g., Dienes' blocks) and the mathematical concept that they are supposed to represent (e.g., place-value). Uttal et al. (1997) suggest that models or concrete manipulatives can be seen in two ways: as objects in their own rights and as representations of something else. The more children treat the materials as objects, the less likely it is that they will discern underlying mathematical concepts. Boulton-Lewis (1999) make a similar point. She suggests that what is really needed is for children to be very familiar with the objects so that the focus of activity is on deepening mathematical understanding rather than on the features of the materials. In other words, children should have ample opportunity to explore through free play the full extent of the materials prior to mathematical discussions. According to Perry and Dockett (2008), the RME interpretation of modeling where models *of* become models *for* mathematical reasoning is preferable (see Chapter 2, Section: *Emphasis on Mathematical Modeling*).

Problem-Solving

Although problem-solving is accorded a central role in the PSMC, it continues to be an area in which children in Ireland underperform (see Report No. 17, Introduction, Section: *Performance Context*). As argued above, much of this rests on the fact that problem-solving is often used as a means of practising acquired skills rather than a context in which to learn mathematics. In relation to this, Hiebert et al. (1996, p. 12) talk about the need to make the subject problematic:

> *Allowing the subject to be problematic means allowing students to wonder why things are, to inquire, to search for solutions, and to resolve incongruities. It means that both curriculum and instruction should begin with problems, dilemmas and questions for students. We do not mean 'problematic' to mean that students should become frustrated and find the subject overly difficult. Rather we use 'problematic' in the sense that students should be allowed and encouraged to problematize what they study, to define problems that elicit their curiosities and sense-making skills.* (p. 12)

They suggest that three kinds of understanding remain ('residue') after a problem is solved: insights into the structure of mathematics, strategies for solving problems, and dispositions towards mathematics. In other words, through engaging in problem-solving, children not only learn problem-solving strategies but also deepen their understanding of mathematics. While play, modeling activities, project work as well as open-ended tasks and other practices discussed in Chapter 2 can be used as contexts for problem-solving, all topics should be introduced to children as 'problematic'. For example, the addition or subtraction of two-digit numbers can be explored via a problem where children are encouraged to construct non-standard algorithms that reflect their developed understanding of place-value (e.g., Cobb, Yackel, & Wood, 1992; Fosnot & Dolk, 2001).

Connecting

The notion of 'connections' in mathematics relates both to those that exist: (i) within and between different content areas in mathematics (e.g., within number or between number and measurement); (ii) between mathematics learning and learning in other areas; and (iii) between mathematics and the context within which a child lives, works or plays (Perry & Dockett, 2008). The idea of connections within mathematics receives considerable treatment in the US NRC (2009) report where it is stated that 'every mathematical idea is embedded in a long chain of related ideas' (p. 48). Johanning (2010) proposes that, in order to build a coherent curriculum and to foster connections, the big ideas from one topic must be built on in others so that children are given the opportunity to use familiar concepts in new settings.

We can see that these processes are strongly interconnected and that they are integral to the development of a mathematics-learning community. They should characterise and be promoted through math talk, that is, children engaging in reasoning, argumentation, justification etc. While we have suggested both in Report No. 17 and earlier in this chapter that a rebalancing of the focus on processes compared with content is required in a revised curriculum, that is not to suggest that content is unimportant. Below we give an overview of the content areas that are found in mathematics curricula for young children internationally, although the labels and levels of emphasis may vary.

Content Areas

In the PSMC, six strands are specified for children in infant classes – Early Mathematical Activities, Number, Algebra, Shape and Space, Measures and Data. For children at higher class levels, the last five of these comprise the content areas. We advise that these five continue to be the broad areas of content in the revised curriculum. However, they should be explicated in ways that reflect current research, and developments in curriculum structure and design. The strand units of Early Mathematical Activities, i.e., Classifiying, Matching, Comparing and Ordering, are now generally addressed within each of the other content areas (e.g., NRC, 2009; Sarama & Clements, 2009; van den Heuvel-Panhuizen, 2008).

In the NRC (2001) report, much attention is given to the domain of number which the authors contend lies at the heart of other strands. However, they emphasise the need to develop mathematical proficiency across all strands of the curriculum:

> *Students need to learn to make and interpret measurements and to engage in geometric reasoning. They also need to gather, describe, analyze, and interpret data and to use elementary concepts from probability. Instruction that emphasizes more than a single strand of proficiency has been shown to enhance students' learning about space and measure and shows considerable promise for helping students learn about data and chance.* (p. 8)

There follows a brief account of each of the five content domains referred to above. While an in-depth treatment of each content area is beyond the scope of this report, some important emphases that need to be taken into consideration in a redevelopment of the mathematics curriculum are identified.

Number

As mentioned in Report No. 17, emphasis needs to be placed on the development of 'number sense', described by Anghileri (2000) as follows:

> *It is not only effort that gives some children a facility with numbers, but an awareness of the relationships that enable them to interpret new problems in terms of results they remember. Children who have this awareness and the ability to work flexibly to solve number problems are said to have a 'feel' for numbers or 'number sense'. What characterises children with 'number sense' is their ability to make generalizations about the patterns and processes they have met and to link new information to their existing knowledge.* (p. 1)

Sarama and Clements (2009) identify the components of number sense as composing and decomposing numbers, recognising the relative magnitude of numbers, using benchmarks, linking representations, understanding the effects of operations, inventing strategies, estimating, and possessing a disposition toward making sense of numbers. Among the 'big' ideas about number that are considered important for the 3- to 8-year-old children are counting, comparing, unitising, grouping, partitioning, and composing.

Arising from a review of the literature, Dunphy (2007) presented the following framework reflecting key aspects of number sense as it relates to 4-year-old children:

- pleasure and interest in numbers (disposition)
- understandings of some of the purposes of numbers (as derived from everyday experiences)
- quantitative thinking (e.g., counting, relating numbers to other numbers, subitising, estimating), and
- awareness and understanding of written numerals (based on interactions about numerals).

There are different approaches to specifying details of the content area of number. Van den Heuvel-Panhuizen (2008, p. 11) concentrates on calculations with whole numbers. The crucial developmental steps that children aged between 2 and 8 take are identified by means of reference points (Intermediate Attainment Targets). She argues that this approach offers the possibility for teachers of '…a helicopter view, the possibility of grasping, in a few large-scale steps, the course of development that takes place'.

The NRC (2009) report identifies three interrelated aspects of early number including whole number, relations, and operations. In relation to each of these, a sequence of milestones for children aged 2–7 are identified. These correspond to the more detailed specification offered by Sarama and Clements (2009) who provide a comprehensive overview of the various elements of content in the number strand. These comprise of:

- quantity, number and subitising
- verbal and object counting
- comparing, ordering and estimating
- arithmetic: early addition and subtraction and counting strategies
- arithmetic: composition of number, place value, and multi-digit addition and subtraction.

For each of these, the authors provide developmental progressions, linked to age. While we see the linking of critical concepts with ages as greatly problematic (see Report No. 17, Chapter 5, Section: *Recognising Developmental Variation*), nonetheless the developmental progressions suggest important concepts that children need to develop.

Some of the key emphases for counting, as deduced from *A Developmental Progression for Counting* (Sarama & Clements, 2009, pp. 73–79), but also informed by van den Heuvel-Panhuizen (2008) and the NRC (2009) report are as follows:

- verbal counting
- making 1–1 correspondence between items and number words (touch counting)
- meaningful (object) counting of small groups (linear)
- answering the question 'How many?'
- meaningful counting of small groups (random arrangement)
- recognising the purposes for which counting is useful
- using fingers to symbolise
- learning the order of counting words beyond 10, beyond 20, to 100 and beyond as required
- counting from a specific number
- counting backwards
- skip counting
- counting imaginary objects.

Children progressively extend the range of their counting. In doing so, they demonstrate increasing interest, focus and effort. As children learn to count (and this learning continues right across the age span 3–8 years), they draw on the other main quantification strategy of subitising. They also make connections to other developing concepts, processes and skills including those related to cardinality and ordinality.

The educator's task is to guide the learning process. The ideas about number indicated above are important. Important too are operations on numbers such as addition and subtraction and procedures for carrying out the operations (NRC, 2005). While international curricula offer different specifications as regards number caps, we conclude that the key issues relate to emphasis on number analysis and key transitions in number analysis (e.g., grouping in tens, addressing 'teens' etc.), rather than number caps per se. Such a position is in keeping with a sociocultural perspective. The various progressions as explicated in the literature offer curriculum designers research-based frameworks which can be drawn on and from which they can extract the important concepts within each aspect of number (and other strands).

Measurement

Measurement is an important mathematical topic because of its applicability to everyday activity, because of its connections with other subject areas and because it can serve as the basis of other content areas in mathematics (Clements, 2003). However, the difficulties inherent in learning measurement concepts should not be underestimated – in particular, measurement differs from number concepts in that it involves the subdivision of continuous quantities into units (Outhred, Mitchelmore, McPhail, & Gould, 2003). A broad outline of appropriate early measurement experiences is as follows (Sarama & Clements, 2009, pp. 274–275):

- encountering, discussing and using appropriate vocabulary for quantity or magnitude of a certain attribute
- comparing two objects directly and recognising equality or inequality
- overcoming perceptual cues and developing the capacity to reason about and measure quantities.

However, each of the topics within measurement presents particular cognitive challenges that need to be addressed. For example, among the challenges that young children encounter in linear measurement are the need to use equal size units, the fact that differing size units lead to different numerical answers (whilst the actual measure is preserved) and the inverse relationship between size of unit and the number of units required for the measure (NRC, 2009). The central concepts in linear measurement are shown in Table 3.1:

Table 3.1: Central Concepts in Linear Measurement

	Idea	Description
Conceptions of unit	Iteration	A subdivision of a length is translated to obtain a measure.
	Identical unit	Each subdivision is identical.
	Tiling	Units fill the space.
	Partition	Units can be partitioned.
	Additivity	Measures are additive, so that a measure of 10 units can be thought of as a composition of 8 and 2, and so on.
Conceptions of scale	Zero-point	Any point can serve as the origin or zero-point on the scale.
	Precision	The choice of units in relation to the object determines the relative precision of the measure. All measurement is inherently approximate.

Source: Lehrer, Jaslow, & Curtis, 2003, p. 102

Sarama and Clements (2009) draw the following conclusions about geometric measurement:

- While it is generally assumed that children learn length first, then area and then volume, this sequencing only applies to the 'spatial structuring' aspects of these measures, e.g., in order to cover a two-dimensional space (area) with units, the child needs to understand the covering of a one-dimensional space (length). In particular, even older primary school children can find 'packing' volume quite challenging. However, other aspects of these measures can develop in parallel (e.g., using a container to measure liquid volume).
- Although there is evidence that young children can develop ideas about attributes such as angle and area from an early age, there is little research to support the investment of time in these topics rather than others.
- While children first develop ideas about measuring different attributes, it takes both time and high-quality educational experiences for them to generalise ideas about measurement across attributes.

There is also evidence that young children can develop concepts of non-geometric measurement such as weight (e.g., Cheeseman, McDonough, & Ferguson, 2012) and time (e.g., Kamii & Long, 2003), although the need to make connections to children's everyday lives and to present stimulating contexts for the learning of each of these topics is stressed. Across all measurement topics, there is some debate about the merits of starting with non-standard units and not with standard measuring devices (e.g., Cheeseman, et al., 2012; Stephan & Clements, 2003). In the NRC (2009) report, research is cited showing that children are often more successful at measuring (length) using standard rather than non-standard units and devices, that using non-standard units actually detracts from children's understanding of basic measurement concepts, that use of a conventional ruler can support mathematical reasoning about length more effectively than non-standard instruments, and that children often show a preference for standard devices. Such findings point to the need to re-examine the linear progression from non-standard to standard units and devices of measurement in the current PSMC.

Geometry and Spatial Thinking

According to Sarama and Clements (2009), geometry and spatial thinking is the second most important area in mathematics learning for young children after number, not only because geometric concepts are important in their own right but also because they support number and arithmetic concepts and skills. Sarama and Clements consider geometric content from three perspectives: (i) the space in which the child lives, (ii) geometric shapes (2-D and 3-D) and (iii) composition and decomposition of shapes. In their consideration of shapes they refer in particular to the work of Pierre and Dina van Hiele who posited five levels of geometric thinking[5], two of which are relevant to young children:

- Level 0 (Visualisation). The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.
- Level 1 (Analysis). The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established (Burger & Shaughnessy, 1986, p. 31).

The levels were considered by the van Hieles to be sequential, discrete and hierarchical (van Hiele, 1959/1984). However, this static view of the levels has been disputed (e.g., Burger & Shaughnessy, 1986). Gutiérrez, Jaime and Fortuny (1991) propose that students use different levels of reasoning depending on the problem to be solved and that there are degrees of acquisition within each level. Clements, Swaminathan, Hannibal and Sarama (1999) maintain that a pre-cognitive level exists before the visual level where children cannot distinguish (2-D) shapes such as circles, rectangles and triangles from non-exemplars of classes of these shapes. According to Clements et al., these children 'are in *transition to*, instead of *at*, the visual level' (p. 205, italics in original). They also

5 The van Hiele model of geometric thinking was developed by Dina van Hiele-Geldof and Pierre van Hiele who completed their doctoral theses on the subject at the University of Utrecht, The Netherlands in 1957. The English translations of the major works of the couple brought the model to the attention of US scholars in the 1970s (Crowley, 1987).

advocate a renaming of the visualisation level as ‘syncretic’, since a level does not consist of ‘pure’ forms of knowledge – for example, visualisation includes both visual/imagistic knowledge and declarative knowledge (‘knowing what’).

Sarama and Clements (2009) provide detailed learning paths for children from birth to 8 years in the following aspects of shape and space:

- spatial thinking (with separate paths for spatial orientation and spatial visualisation and imagery)
- shape
- composition of 2-D shapes
- composition of 3-D shapes
- embedded geometric figures.

A feature of their learning paths is the use of descriptive terms to describe the processes of children at different levels. Hence, in the case of spatial visualisation, we find simple sliders (who can move shapes to a location), simple turners (who can mentally turn objects in easy tasks), beginning sliders, flippers and turners (who can use correct motions, but not always accurately), more advanced sliders, flippers and turners (who can perform slides and flips, using manipulates, and make turns of 45, 90 and 180 degrees), diagonal movers (who can perform diagonal slides and flips), and mental movers (who can predict results of moving shapes using mental images).

Another description of the progression of 3- to 5-year-olds in geometry/spatial thinking can be found in the NRC (2009) report, which provides learning paths for space and shape in two dimensions, and in three, with each learning path focusing on describing and constructing objects, spatial relations, and compositions and decompositions. The NRC report highlights the importance of providing young children with substantial experience of shape and space, and warns that, if the shape categories that children experience are limited, so will their concepts of shapes. One implication of this is that children need to encounter ‘rich and varied examples and non-examples, and discussions about shapes and their characteristics’ (p. 192). The NRC (2009) report also suggests a range of activities designed to support children’s development of spatial thinking.

The learning paths provided by Sarama and Clements and by the NRC are quite detailed, possibly because the authors feel that teachers in the United States may themselves have limited understanding of geometry and spatial reasoning, and hence might benefit from a high level of detail. An alternative approach is evident in the work of van den Huevel-Panhuizen and Buys (2008) who provide intermediate learning targets in geometry for children in kindergarten 1 and 2 (junior and senior infants) and grades 1 and 2 (first and second classes). The intermediate learning targets are brief narrative descriptions of mathematical processes children at each grade range can be expected to engage in. There are three targets at each grade range – one each covering orienting (describing position in space), constructing, and operating with shapes and figures. The intermediate learning targets are accompanied by descriptions of the associated mathematical reasoning, and of activities that might be presented to children to support their development. Many of the activities are rooted in children's everyday experiences, or are embedded in fictional stories that provide realistic contexts for activities such as map-making.

van den Huevel-Panhuizen and Buys (2008) also note a link between geometry/spatial reasoning and a range of goals of primary education including:

- developing a positive working attitude
- making connections between mathematics and daily life
- making practical applications
- reflecting on one's own mathematical activities
- developing and designing connections, rules, patterns and structures.

They focus, in particular, on the aesthetic value of geometry (making patterns, use of symmetries, discovering structure in nature, developing an eye for geometric elements in art, design and architecture), which, they argue, can contribute to the cultural development of primary school children, as well as developing their mathematical proficiency.

Algebraic Thinking

While there are many perspectives on the nature of algebra and particularly what might constitute algebraic thinking in the early grades (e.g., Cai & Knuth, 2011; Kaput, Carraher, & Blanton, 2008), Kieran (2011, p. 581) suggests the following themes as those that predominate recent research literature on the subject:

- Thinking about the general in the particular – this idea stems in particular from the work of John Mason (mentioned in the section on '*Generalising*' above).
- Thinking rule-wise about patterns – this concerns not just determining a commonality in a sequence but extending the rule to indeterminate quantities.
- Thinking relationally about quantity, number and number operations – this involves seeing numbers and number operations in terms of their inherent structural relations (e.g., $8 + 5 = 10 + 3$).
- Thinking representationally about the relations in problem situations – this pertains to using a variety of representations (e.g., context, manipulatives, drawings) to visualise a problem situation.
- Thinking conceptually about the procedural – this approach to mathematical procedures implies a focus on rich mathematical connections, generalities and relationships that emanate from the procedure.

- Anticipating, conjecturing, and justifying – in particular this concerns the development of a classroom culture where questions are used by the teacher to move students forward in their thinking; where students explain and justify their reasoning and where they delve into challenging mathematical ideas.
- Gesturing, visualising, and languaging – young children draw on a variety of ways – visual, aural, motor senses – to express pattern regularity (e.g., they might use gesture and /or words to signify a 'non-present' element of the pattern).

What is apparent from this list is that many of the characteristics of algebraic thinking are analogous to the processes (e.g., communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting) described earlier. From this perspective, algebraic thinking serves to give a deeper treatment to other content domains. Bastable and Schifter (2008) put it like this:

> *When the arithmetic classroom environment is designed to follow children's thinking and provides elementary students with the opportunity to pursue their own questions, they display interest and ability in formulating and testing generalizations. Although these students do not, of course, use conventional algebraic symbols to express their ideas, the kinds of arguments they pose and the kinds of reasoning they display have parallels in formal algebra.* (p. 165)

Thus the infusion of algebraic thinking across the mathematics curriculum would facilitate the development of the processes.

In terms of pattern work, children should be given the opportunity to explore a wide range of materials. Their attention can be drawn to the many patterns in nature and in their everyday environment. Initially in preschool, children should explore sequences since the ability to recognise sequences is important in pattern work. When children recognise that repeating sequences form a pattern, they can begin to organise their pattern making. This can focus on different attributes, e.g., size, colour, shape, orientation etc. They can deal with both pattern making and pattern perception but appear, initially, to find it easier to talk about the characteristics of patterns that they have created themselves than to discuss those created by others (Garrick, Threlfall, & Orton, 1999). As they grow older, there needs to be a focus not only on creating and recognising patterns but also on increasing the complexity of patterns (Threlfall, 1999). Later children can move towards describing a pattern numerically.

In order to encapsulate the breadth of the domain of algebraic thinking, Cooper and Warren (2011) suggest a framework for curriculum that encompasses (i) pattern and functions, (ii) equivalence and equations and (iii) arithmetic generalisation. In the revised curriculum, consideration could be given to explicating these in two strands, i.e., Algebra and Pattern, and Number (e.g., ACARA, 2009).

Data and Chance

Data is the domain that receives least attention in research on mathematics education in the early years (Clarke, 2001; Sarama & Clements, 2009). Sarama and Clements (2009) suggest that in order for children to understand data analysis they must learn concepts of 'expectation' (e.g., averages, probability) and 'variation' (uncertainty, spread of values). Jones et al. (2000) formulated a framework for characterising children's statistical thinking. The four constructs in the framework are 'describing', 'organising', 'representing' and 'analysing and interpreting' data. For each construct there are four thinking levels – idiosyncratic, transitional, quantitative and analytical –on a continuum. In a small scale study of 20 children from grades 1 – 5 (US), they found that children in grades 1 and 2 typically exhibited thinking at level 1 (idiosyncratic) or level 2 (transitional)[6]. They also found lowest levels of thinking on the 'analysing and interpreting' construct, a finding they attribute to possible poor focus on this construct in classroom activities. Leavy (2008) suggests that children's ownership of a statistical problem is a critical factor in developing their statistical reasoning beyond level 1. In this regard, it is interesting that in the recently developed Australian mathematics curriculum (ACARA, 2009), the content domain is termed 'statistics and probability' rather than 'data and chance' in order to emphasise the need for children to interpret and analyse as well as represent and summarise data. While probabilistic reasoning ('chance') has not traditionally featured in mathematics curricula for children aged 3–8 years because of the cognitive challenges that it poses, it now tends to be included from kindergarten on. The emphasis is on language development, e.g., 'might', 'maybe' and the need to ground understanding in children's everyday lives (e.g., Metz, 1998).

Content Areas and Curriculum Presentation

Linkages

Although there is generally broad agreement on the content domains listed above, in recent curricula there has been a tendency to amalgamate some of the domains. For example, in the study of measurement and geometry there are some considerable overlaps. Such overlaps also exist between measurement and data, number and algebra etc. The domains listed for US Common Core States Standards for Mathematics (CCSM) are as follows (White & Dauksas, 2012):

- Operations and Algebraic thinking
- Number and Operations in Base ten
- Measurement and Data
- Geometry.

6 In representing data, for example, a child at thinking level 1 would produce an invalid or idiosyncratic display of a data set while a child at thinking level 2 would produce a display that is partially valid.

The content strands in the Australian curriculum (ACARA, 2009) are:

- Number and Algebra
- Measurement and Geometry
- Statistics and Probability
- Geometry and Measure.

As mentioned earlier, consideration also needs to be given to the idea of an integrated curriculum in which mathematical concepts and skills are developed across a class-level rather than repeating ideas from year to year (Johanning, 2010). For example, certain content areas might be emphasised at a particular class level but over an extended period, e.g., two years, all content areas would receive attention. This allows for more in-depth exploration of key topics.

Learning Outcomes

In the revised Dutch curriculum, the goals describe opportunities to learn rather than intended competencies or content objectives. This subtle shift has important consequences. If mathematical content is framed as a list of competencies, the result is narrow and basic since the content has to apply to all students. Opportunities to learn, on the other hand, give more scope to describe what is considered important for students to learn – 'whether they actually will learn this…cannot be fixed' (van den Heuvel-Panhuizen & Wijers, 2005, p.293). This places more focus on learning than on attainment. In this regard, the call in the *National Strategy to Improve Literacy and Numeracy* (DES, 2011a) for the use of learning outcomes as opposed to content objectives is welcome. As described in Report No. 17 (Chapter 4, *Table 4.1. Specifying Goals: Different Approaches*), in the US *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (NCTM, 2006), critical ideas are broken down into transitions that indicate shifts in mathematical reasoning. These narrative descriptors, together with goals and learning paths, contribute to the formulation of learning outcomes. Such an approach might provide a basis for structuring the curriculum at content level, with the content-level descriptors providing a basis for identifying learning outcomes. Figure 3.1 shows an emerging curriculum model highlighting how the relationships between the different elements may be conceptualised.

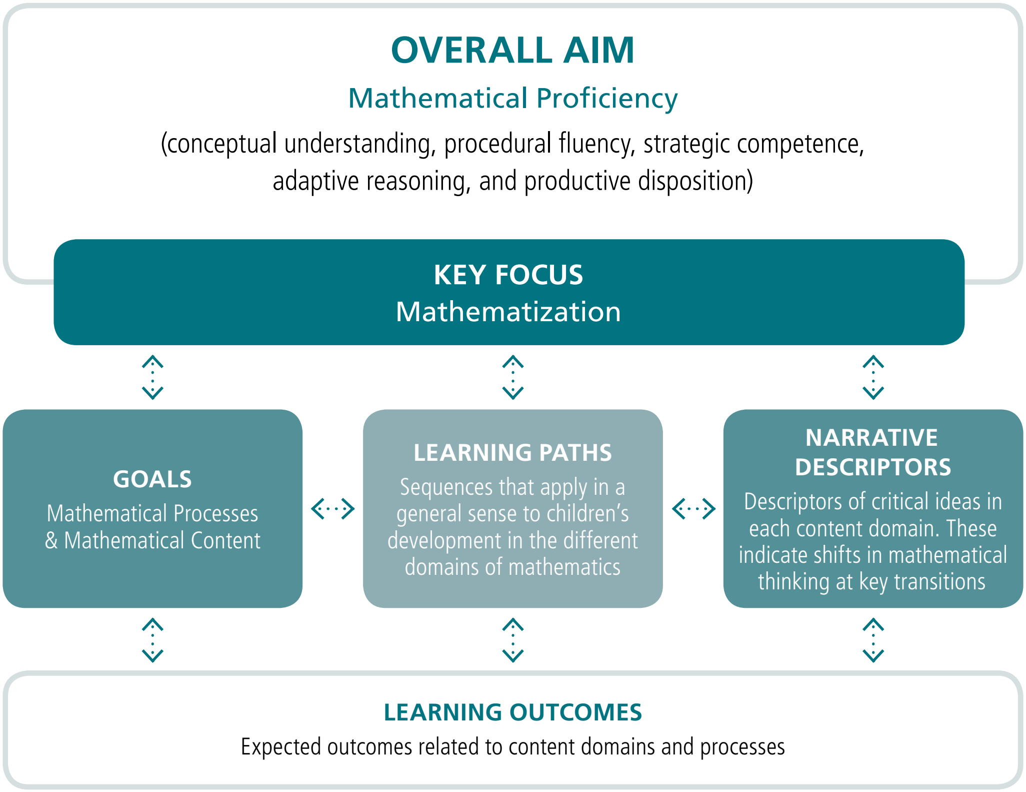

Figure 3.1: Emerging Curriculum Model

Conclusion

In this chapter we have given consideration to ways in which the mathematics curriculum for 3- to 8-year-old children might be developed. In particular, we have argued for the development of a coherent curriculum where there is close alignment between the aim of mathematical proficiency and goals related to processes and content. Engagement with processes of communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting serves to deepen children's mathematical learning. The content areas related to Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance – in whatever way they are labelled – constitute the mathematical knowledge with which children should engage.

The key messages arising from this chapter are as follows:

- Goals of the curriculum should relate both to processes and content.
- The processes of mathematization, that is, communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting, should be foregrounded.
- In line with the principle of 'mathematics for all', each of the five domains – Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance – should be given appropriate attention.
- While critical ideas in each content domain need to be explicated, over-specification or age-specification should be avoided.
- Narrative descriptors of critical ideas indicating shifts in children's mathematical reasoning are potentially useful for teachers.
- Learning outcomes, derived from narrative descriptors, are a preferred alternative to content objectives.

CHAPTER 4

Curricular Issues

In this chapter we consider a number of curricular issues related to curriculum implementation and effective mathematics pedagogy. These include provision of an equitable mathematics curriculum that is inclusive of all children; early intervention; allocation of time to teaching mathematics; and integration of mathematics across the curriculum. In considering equity issues, we discuss the needs of exceptional children, including those with intellectual and developmental difficulties, and children with mathematical talent. We also focus on children in culturally-diverse contexts including English language learners, children learning mathematics through Irish, and children living in disadvantaged circumstances.

An Equitable Curriculum

The vision of equity in a curriculum 'challenges teachers to raise expectations for the mathematical learning of all students and to provide instruction that responds to students' prior knowledge, academic strengths, and individual interests' (Lloyd & Pitts Bannister, 2010, p. 324). In Report No. 17, we discussed how 'mathematics for all' implies a pedagogy that is culturally sensitive and takes account of individuals' ways of interpreting and making sense of mathematics. Furthermore, we highlighted that exceptional children (those with developmental disabilities or who are especially talented at mathematics) do not require distinctive teaching approaches but should have their individual needs met (see Report No. 17, Chapter 7, Section: *Exceptional Children*).

The view of mathematics that is integral to this report and to Report No. 17 means that the mathematics curriculum is not neutral and objective but is mediated by culture. An implication of this is that the mathematics curriculum and pedagogy have to take account of learners' interests, backgrounds and ways of knowing. Among the ways that Ladson-Billings (1995) identifies as useful in fostering a multicultural curriculum and pedagogy that embraces the needs of all students are the following:

- The importance of treating all children as if they *already* have knowledge and experience that can be used as a foundation for teaching.
- The creation of a learning environment that allows children to move from what they do not know to what they do know.
- A focus on high-quality mathematics learning rather than on 'busy' work.
- The provision of challenging tasks to *all* children.

- The development of in-depth knowledge of children and subject matter.
- The fostering of strong teacher-child relationships.

These pedagogical features are also consistent with those identified in Chapter 2. In this regard, the strategies used to cater for the diverse needs of learners constitute 'good teaching', helping children to 'see mathematics as a human endeavour done by real people to serve real needs and interests' (Wiest, 2001, p. 22). A key issue in developing an inclusive classroom is the philosophical orientation of the teacher. This includes adherence to the belief that 'it is helpful to view difficulties in learning as problems for teachers to solve rather than problems within learners' (Florian & Linklater, 2010, p. 371). In addition, it demands that in schools there is an 'ethic of everybody: teachers have both the opportunity and responsibility to work to enhance the learning of all' (Florian & Linklater, 2010, p. 372).

Exceptional Children

As noted in Report No. 17 (see Chapter 7, Section: *Exceptional Children*), Kirk, Gallagher, Coleman and Anastasiow (2012) define as 'exceptional' a child who differs from the 'typical' child in (i) mental characteristics, (ii) sensory abilities, (iii) communication abilities, (iv) behaviour and emotional development, and/or (v) physical characteristics. The term includes both children with developmental delays and those with gifts and talents.

Children with Intellectual and Developmental Difficulties

In the literature on inclusive strategies, the question arises concerning the extent to which specialised approaches are needed for some children with special educational needs. In a review of pedagogies for inclusion, Lewis and Norwich (2005) proposed the notion of continua of common teaching approaches that can be subject to various degrees of intensity depending on individual need. However, they also state that 'in advocating a position that assumes continua of common pedagogic strategies based on unique individual differences, we are not ignoring the possibility that teaching geared to pupils with learning difficulties might be inappropriate for average or high attaining pupils' (p. 6). An example of intensification of a common teaching approach is that used by Staves (2001) in teaching counting to children with a moderate general learning disability. He suggests, among other strategies, those of applying different facets of attention such as using a small torch to point to objects in turn; increasing the emphasis of motion, rhythm or pressure when reaching the last object when guiding a child's finger; and varying the intensity of volume and vocal tones for the last number in a sequence (Staves, 2001).

Children with Hearing Impairment

Recent research has demonstrated that 'deaf children have different knowledge, learning styles and problem-solving strategies than hearing children. Teachers need to know how their deaf students think and learn if they are to accommodate their needs and utilise their strengths' (Marschark & Spencer, 2009, p. 210). Recommendations for deaf and hard-of-hearing children include recognising their visual-spatial orientation, which they do not always apply, and their relative lack of confidence in problem-solving. Children with hearing loss 'face special difficulties when needing to relate multiple bits of information and to identify relationships' (Marschark & Spencer, 2009, p. 140). They suggest that 'it is clear that modifications in curricula and in teaching strategies are required if deaf and hard-of-hearing students are to develop to their potential in the important areas of maths and concepts' (p. 140). Interventions which have shown promise include those which focus on building problem-solving skills through producing schematic illustrations emphasising visual-spatial over verbal activities (Nunes, 2004).

Children with Visual Impairment

Access to a mathematics curriculum for children with visual impairment often hinges on specialist teacher knowledge of the unique aspects of mathematics education for such children. This includes use of calculation with abacus or braillewriter, talking calculator, concrete materials and tactile displays and teaching of the Nemeth Code (Kapperman, Heinze, & Sticken, 2000). A focus on mathematical language and its accuracy by the educator is also stressed.

Children with Autistic Spectrum Disorders

- Sensitivity to the individual needs in mathematics of children with autistic spectrum disorders (ASD) is also essential for teachers. Such children may not join in class counting activities and may find *counting on* difficult. In guidance to schools, the Department for Education and Skills (2001, p. 1) in England suggests that children who find imaginative play and play with others difficult may not have built up a wide store of mathematical concepts through engagement in such activities. Therefore a wide range of structured contexts must be provided to support the development of concepts and language. Children with ASD find some illustrations confusing and teachers may need to explain these using appropriate language.

Mathematically-talented Children

Mathematically-talented children are those who have very high levels of competence in mathematics, and can solve mathematical problems that could be considered advanced for their class level. One way in which the needs of these children might be met is through the use of 'tiered assignments' (Fiore, 2012) – that is, parallel tasks that have different levels of depth, complexity and abstractness, and different support elements or guidance, though all children work towards the same general learning outcomes. For mathematically-talented children, tasks may be differentiated by including more complex numbers, by adding obstacles to the solution process, by requiring children to engage in novel solution strategies, or by requiring them to use particular representations.

A related approach, 'curriculum compacting' (Reis, Burns, & Renzulli, 1992; Renzulli, 1994) involves (i) defining the goals and outcomes of a particular unit or segment of instruction; (ii) determining and documenting which students have already learned most or all of a specific set of learning outcomes; and (iii) providing extension strategies for material already learned through the use of instructional options that enable a more challenging and productive use of the child's time. There are other strategies for presenting the curriculum to mathematically-talented children including:

- Introducing mathematical ideas beyond those typically addressed for their age group or class.
- Developing the self-assessment and self-regulation skills needed for planning, self-assessment, monitoring, and evaluating learning activities.
- Providing a wider range of open-ended investigatory tasks.
- Providing tasks that are of interest such as those involving very large numbers, abstract mathematical explorations, and applications of mathematical ideas in a broader range of contexts.

In this view, a child's prior knowledge and strengths should guide the selection and implementation of tasks, rather than a mainly age- or grade-level focused curriculum. All of these approaches recognise that mathematically-talented children should be supported in deepening their understanding of the existing curriculum rather than being provided with an alternative one.

Children in Culturally Diverse Contexts

In this section, we consider the curricular needs of children in culturally diverse contexts, including English language learners, children learning mathematics through Irish, and children in socio-economically disadvantaged contexts.

English Language Learners

Much of the research on developing children's mathematical discourse has been conducted in settings involving young English language learners: children whose first language differs from the language of instruction. For example, Hufferd-Ackles et al. (2004) demonstrated how teachers of English language learners in third grade successfully made the transition from traditional, teacher-led pedagogy to a math talk community over the course of a school year, albeit with a modified curriculum and weekly support from a university-based mentor. Prediger, Clarkson and Bose (2012) identify three broad pedagogical strategies that are relevant to teaching mathematics to language learners. First, they highlight code switching as an important resource, in that it provides a comfortable and flexible mode of communication, and enables simultaneous learning of language and mathematics. Second, they emphasise a need to support young children in making the transition from the everyday language to a technical or mathematical register, which, they argue, can enhance children's understanding of mathematical concepts and ideas. In doing so, they note that children may have registers for everyday language, school language, and mathematical/technical language in both their home language and in the language of instruction. Third, they stress a need to facilitate transitions between different mathematical representations – for example, between pictorial and symbolic representations, or verbal and written representations – in order to build conceptual understanding. For example, they argue that a pictorial representation can ease the language burden during initial presentation of a topic or problem, and that the emphasis can proceed to language after the underlying concept has been learned.

The literature acknowledges that second-language learners can encounter particular challenges in math talk learning communities (Chapin et al., 2009). Without an understanding of the relevant vocabulary, syntax and grammar, such learners may be prevented from demonstrating the depth of their understanding and from engaging productively in various learning activities such as explaining solution processes, describing conjectures, proving conclusions and presenting arguments, in both verbal and written contexts. Moschkovich (1999) emphasises that, when the goal is supporting children's engagement in mathematical discussion, listening and responding to the quality of mathematical discourse is as important as focusing on children's language proficiency, and aspects of language that relate to mathematics can be attended to from within a content-focused discussion. According to Moschkovich, the instructional strategies that might be used to attend to language in mathematics content contexts include:

- using several expressions for the same concept
- using gestures and objects to clarify meaning
- accepting and building on children's responses
- re-voicing children's statements using more technical (mathematical) terms
- focusing on mathematical content and argumentation.

Children Learning Mathematics through Irish

The outcomes of the *2010 National Assessments of English Reading and Mathematics in Irish-Medium Schools* point to challenges that teachers in Gaelscoileanna encounter in providing instruction in mathematics through the medium of Irish (see Gilleece, Shiel, Clerkin, & Millar, 2012). Whereas performance in mathematics was ahead of the national average at second class, it did not differ significantly from the national average at sixth class, with higher-achieving children failing to maintain the advantage enjoyed by their counterparts in second class. At sixth class, performance was above national standards on procedural aspects of mathematics, but not on reasoning and problem-solving. Teachers of 20% of children in second class in Gaelscoilenna, and teachers of 80% of children in sixth class reported teaching mathematics in both English and Irish. Teachers attributed this shift in emphasis to a need to prepare children for learning mathematics in English-medium post-primary schools and to support children who might struggle to acquire important mathematical concepts through Gaeilge.

The *2010 National Assessment of Mathematics* did not gather information on the extent to which children were engaged in math-talk learning communities, such as those envisaged in the current report. However, a study by Ryan (2011) indicates that some children in Gaelscoileanna experience difficulty in using mathematical terminology and in articulating higher-order concepts and reasoning processes as they engage in problem-solving in small group contexts. The outcomes of the national assessment and of Ryan's study suggest a need for Gaelscoileanna to pay particular attention to promoting opportunities for children to develop and use mathematical language as they engage in reasoning and problem-solving processes, in mathematics classes, and, more generally, across the curriculum.

While the development of mathematical language in immersion settings such as those found in Irish-medium schools is undoubtedly complex and may relate to a range of policy, teacher and child factors, the literature suggests that opportunities to generate, discuss, explain and justify mathematical ideas benefit all children in their development of mathematical language, regardless of the language in which instruction is provided (Barwell, Barton, & Setati, 2007; Gutiérrez, Sengupta-Irving, & Dieckmann, 2010; Ryan, 2011).

Another group of children who may learn mathematics through Irish are those attending schools in Gaeltacht areas. Gilleece et al. (2012) showed that, while children in second class in Gaeltacht schools achieved a mean score that was not significantly different from the national average, children in sixth class achieved a significantly higher average score. Importantly, Gilleece et al. reported that 45% of children in second class in Gaeltacht schools were taught mathematics in Irish only, while the remainder were taught through a combination of English and Irish. By sixth class, 50% were taught through Irish, and 50% through a combination of English and Irish. These data point to variation in competence in Irish among children in Gaeltacht schools. They also reflect the efforts of schools and teachers to adjust their use of language to take that range into account. In terms of curriculum development, it is important to support children in Gaeltacht schools as much as possible in accessing the full mathematics curriculum in the Irish language. We suggest that a strong focus on mathematical discourse from the beginning of children's schooling can play a significant role in achieving this (see Report No. 17, Chapter 3, Section: *The Role of Language in Developing Mathematical Knowledge*; this report, Chapter 2, Section: *Promotion of Math Talk*).

Children in Socioeconomically Disadvantaged Contexts

As noted in Report No. 17 (Chapter 3, *Variation in Language Skills and Impact on Mathematics*), less advantaged children, prior to attending school, typically use the same informal strategies to solve addition and subtraction problems, they perform at about the same level as more advantaged children on non-verbal addition and subtraction problems, and they exhibit few if any differences in the everyday mathematics they employ in free play (Ginsburg, Lee, & Boyd, 2008). Hence, the challenge for teachers is to support less-advantaged children to acquire mathematical language and metacognition – the ability to express and justify their own mathematical thinking – as early in their development as possible.

Children living in less-advantaged circumstances may struggle to participate in mathematics learning contexts that emphasise mathematical discourse as a learning tool (e.g., Lubienski, 2002; Anthony & Walshaw, 2007). Given the key role of language and discourse in mathematics learning, such concerns reinforce the need for intensive instructional support for less advantaged children from an early age (e.g., NRC, 2009) to prepare them to meet the language demands of discourse-based mathematics teaching and learning. Specific strategies include:

- frequent exposure to mathematical language, in both formal and informal contexts (e.g., Klibanoff et al., 2006)
- intentional teaching of mathematical vocabulary using multi-modal methods, with attention to categorisation and associations between related concepts (e.g., Neuman, Newman, & Dwyer, 2011)
- planned opportunities to use language in mathematical problem-solving contexts with varying degrees of structure
- planned opportunities to use mathematical language across a range of curriculum areas (see this volume, Chapter 2, Section: *Practices in Integrative Contexts*).

Although there is evidence that children in the urban dimension of the *School Support Programme* (SSP) under DEIS has had some impact on mathematics achievement at second, third and sixth classes between 2007 and 2010 (Weir, Archer, O'Flaherty, & Gilleece 2011), average gain scores are typically small (2–3 standard score points on scales with a mean of 100, and a standard deviation of 15), and it is unclear whether gains are attributable to the SSP as a whole or to one or more of its constituent programmes, such as *Maths Recovery*. The observation that children in DEIS schools continue to lag behind children in non-DEIS schools in mathematics and in other areas of the curriculum (e.g., Eivers et al., 2010) points to a need to intensify the set of mathematics interventions in DEIS schools. While some of the impetus for change will come from the redeveloped curriculum, it is likely that a broader suite of interventions will also be needed. These may include:

- allocation of additional time for mathematics teaching and learning
- ongoing continuous professional development in mathematics for teachers
- affirmation of strategies and programmes that are working effectively to improve children's mathematics achievement
- access to and support in maintaining and using a broad range of resources for teaching mathematics, including digital learning resources
- intensive learning support interventions for children who are most at-risk that are integrated with classroom instruction
- an emphasis on formative assessment, to complement the strong emphasis on summative assessment in DEIS schools in recent years.

Early Intervention

In a review of the literature on intervention in mathematics, Dowker (2004) makes the case for early intervention, arguing that 'research strongly supports the view that children's arithmetical difficulties are highly susceptible to intervention' (p. 42). In terms of the type of intervention, she emphasises the benefits of individualised instruction:

> *Moreover, individualized work with children who are falling behind in arithmetic has a significant impact on their performance. The amount of time given to such individualized work does not, in many cases, need to be very large to be effective.* (p. 43)

Dowker's (2004, 2009) overviews suggest that targeted interventions based on a diagnostic assessment (see Report No. 17, Chapter 6, Section: *Diagnostic and Summative Assessment*) of the strengths and needs of the child in relation to mathematics can be very beneficial and should be a feature of any support system put in place to address low achievement in mathematics. The crucial issue here is not so much the allocation of additional time but rather one of more focused teaching approaches, and clarity about the nature of the learning to be addressed.

There is a key role for the learning support/resource teacher in terms of supporting a prevention and early intervention policy in schools. The *Learning-Support Guidelines* (DES, 2000), while highlighting the role of intensive prevention, interpret early intervention as occurring from senior infants and this needs to be revisited. Many schools only implement early intervention in mathematics from first class onwards (Mullan & Travers, 2010). As noted in Report No. 17, we now have a greater understanding of the range of mathematics that very young children can engage in and the diversity in early mathematical knowledge and skills displayed by children starting school. In addition, we know the importance of disposition and how this can be damaged, affecting engagement and participation in mathematics. This necessitates a much earlier role for prevention and early intervention. The learning support/resource teacher can assist the class teacher in identifying children at risk of mathematical difficulties and engage in in-class as well as external support to address their needs (DES, 2005b).

Allocation of Time to Teaching Mathematics

This section considers the allocation of time to teaching mathematics. First, it looks at allocation of time in preschool settings, and then moves on to primary-school settings. The allocation of time for engagement of children in mathematically-related learning, for example, in working with small groups in a preschool setting, or with larger groups in a primary-school classroom, comprises just one of the contexts in which children may encounter mathematics. Children also engage in mathematics during structured play activities (as recommended in *Aistear*, for example), and in cross-curricular contexts (see below). Throughout this section we emphasise that, while allocation of both dedicated time to mathematics and the integration of mathematics with other areas of learning are important, what is paramount is the quality of the pedagogy (see this volume, *Chapter 1*, *Chapter 2*).

Preschool Settings

Literacy is often seen as an over-riding goal in preschool settings, with considerably less time allocated to mathematics or numeracy (e.g., Lee & Ginsburg, 2009; Thomson, Rowe, Underwood, & Peck, 2005). Now, however, in line with an enhanced understanding of how young children develop mathematically, it is recognised that 'children require significant amounts of time to develop the foundational mathematical skills and understandings they have the desire and potential to learn and that they will need for success at school' (NRC, 2009, p. 124). While it is acknowledged that some children can acquire foundational skills at home through spending significant time on focused interaction with family members, it is argued that

> *Even children who learn mathematical ideas at home will benefit from a consistent high-quality program experience in the preschool and kindergarten years. It is therefore critical that sufficient time is devoted to mathematics instruction in preschool programs so that children develop foundational mathematical skills and understandings…Time must be allocated not only for the more formal parts of mathematics instruction and discussions that occur in the whole group or in small groups, but also for children to elaborate and extend their mathematical thinking by exploring, creating, and playing.* (NRC, 2009, p. 124)

An implication of this proposal is that all children should engage in a preschool mathematics programme, in which there is intentional teaching of early mathematics whether in the context of structured whole group or small-group sessions, or in play contexts. Other contexts in which preschool educators can promote mathematical concepts and language include, for example, playing games, reading books with a mathematical theme, using computers, and constructing objects (e.g., block building) (see this volume, Chapter 2, Section: *Practices in Integrative Contexts*). Regardless of the context, however, there is a need for preschool teachers to identify key concepts that children need to learn, and to provide relevant experiences (including materials) that enable children to acquire those concepts. When preschool teachers work with parents to identify opportunities for mathematical development at home, the amount of time in which children attend to mathematical ideas can be increased substantially (NRC, 2009).

Primary School Settings

The PSMC (Government of Ireland, 1999a) suggests that schools allocate a minimum of two hours and fifteen minutes per week to mathematics where there is a short school day for infants, and three hours per week at other class levels. Schools could add discretionary time to this (one hour in the case of infant classes operating with a shorter day, and two hours for other classes), though such time could be allocated to other curricular areas instead.

There have been concerns in Ireland regarding curriculum overload – or the absence of sufficient time to cover all aspects of the curriculum (e.g., NCCA, 2010). One response to such concerns, as they relate to mathematics education, has been the provision by the NCCA of re-presented content objectives for mathematics (NCCA, 2009c), which seek to present content objectives in a format that renders them more navigable, enabling teachers to more easily see links across objectives between junior infants and second class. A broadly similar approach is used in Northern Ireland, where colour codes are employed to show links across learning statements (CCEA, n.d-a.).

The available evidence suggests that most schools typically exceed the minimum time allocations specified by the NCCA. In the *2009 National Assessments of Mathematics and English* (Eivers et al., 2010), teachers reported allocating 3 hours and 45 minutes to mathematics in second class (and 4 hours and 18 minutes in sixth). Following publication of the *National Strategy to Improve Literacy and Numeracy* (DES, 2011a), where concerns were raised about standards in numeracy (and literacy), the DES issued a circular (0056, 2011), which required schools to increase, from January 2012, the allocation of time spent on mathematics by 70 minutes per week to 3 hours and 25 minutes per week for infants with a shorter day, and to 4 hours and 10 minutes per week for children with a full day. The circular stated that this could be achieved through integrating numeracy with other curriculum areas, using discretionary curriculum time for numeracy activities, reallocating time spent on other subjects in the curriculum to numeracy, and delaying the introduction of other curricular areas. Since the 2009 national assessment of mathematics indicated that teachers of second class were, on average, allocating an average of 3 hours and 45 minutes per week to mathematics, the circular essentially indicates that an additional 25 minutes per week on average (or 5 minutes per day) should be added.

An analysis of data on allocation of teaching time in *Growing Up in Ireland* found that teachers in classrooms of 9-year olds (second to fourth classes) allocated 3.7 hours per week to mathematics in 2007–08 (McCoy, Smyth, & Banks, 2012). While 3 hours or less per week were allocated to mathematics teaching in 40% of primary classrooms, allocation was 5 or more hours in one-quarter of classes. Hence, there is considerable variation around average time allocations. McCoy et al. also reported no significant difference in allocated time for mathematics in DEIS and non-DEIS schools. Male teachers reported spending more time teaching mathematics than females, while teachers in Gaelscoileanna allocated significantly less time to teaching mathematics than their counterparts in other school types. More variation was observed across teachers within schools than across schools in terms of the allocation of time to mathematics teaching, suggesting that teachers enjoyed some autonomy in allocation of instructional time. McCoy et al. also found that the allocation of additional time to mathematics (and English) in the *National Strategy to Improve Literacy and Numeracy* might not affect the time allocated to other subjects if the additional time involved teaching literacy and numeracy across curriculum areas.

Finally, the *2011 Trends in International Mathematics and Science Study* (TIMSS) (Mullis, Martin, Foy, & Arora, 2012), in which children in fourth class in Ireland and in 49 other countries/jurisdictions participated, reported considerable variation in the yearly allocation of time to teaching mathematics. In Ireland, children received 150 hours of mathematics teaching (about 4 hours per week), whereas their counterparts in Northern Ireland, who had a significantly higher average mathematics score than Ireland, received 232 hours (Table 4.1). Ireland's average time was also below the international average of 162 hours. However, TIMSS does not show a linear relationship between time allocation and performance, with, for example, children in Japan also performing well ahead of Ireland, even though they allocated the same amount of annual instructional time. This suggests that allocation of time to mathematics teaching is just one of a number of factors that contribute to actual achievement.

Table 4.1: Annual Allocation of Time to Teaching Mathematics and Mean Achievement – Selected Countries at Grade 4 (TIMSS, 2011)

Country	Annual Time (Hours)	Mean Achievement
Northern Ireland	232	562
Singapore	208	606
United States	206	541
Netherlands	195	540
England	188	542
International Average	162	500
Ireland	150	527
Japan	150	585
Finland	139	545

Source: Mullis et al. (2012), Exhibits 1.1 and 8.6

Of course, allocation of additional time to mathematics instruction may not, by itself, lead to increased mathematical proficiency. According to the NRC report (2001):

> *[ ] instruction can best be examined from the perspective of how teachers, students, and content interact in contexts to produce teaching and learning. The effectiveness of mathematics teaching and learning is a function of teachers' knowledge and use of mathematical content, of teachers' attention to and work with students, and of students' engagement in and use of mathematical tasks. Effectiveness depends on enactment, on the mutual and interdependent interaction of the three elements – mathematical content, teacher, students – as instruction unfolds. The quality of instruction depends, for example, on whether teachers select cognitively demanding tasks, plan the lesson by elaborating the mathematics that the students are to learn through those tasks, and allocate sufficient time for the students to engage in and spend time on the tasks.* (pp. 8–9)

Allocation of time to mathematics, and, in particular, children's engagement in meaningful mathematical activities, are important factors associated with mathematical proficiency, but, by themselves, they do not guarantee high levels of proficiency, and a range of other factors that contribute to effective instruction need to be considered (e.g., Scheerens, 2004). Nevertheless, sustained time – whether in preschool or school contexts – is an important pre-condition if children are to engage in mathematization and participate in math-talk learning communities.

Emphasis on Different Mathematics Content Areas

In addition to allocation of overall time to mathematics, the allocation of time to individual mathematics content areas may be important. Teachers in TIMSS 2011 in Ireland indicated that 56% of instructional time in mathematics in fourth class was allocated to Number, 22% to Geometry and Measures (combined), 12% to Data Display, and 10% to other topics (Close, 2013). Close argues that the Irish data reflect an over-emphasis on Number, a domain on which children in Ireland tend to do well in national and international assessments. There is a concomitant under-emphasis on Shape and Space, Measures and associated problem-solving activities, on which children in Ireland tend to do less well. While these data relate to fourth class, they suggest a need to ensure a better balance in time allocation across mathematics content areas, with proportionally less time allocated to Number and procedural activities, and more time allocated to Shape and Space and Measures.

In the next section, we look at how mathematics activities might be integrated into other areas of learning, not only to provide children with additional mathematical experiences, but also to help them to see the relevance of mathematical ideas in a broad range of other contexts.

Mathematics Across the Curriculum

In Chapter 2, we identified several practices that teachers can use to support the development of mathematics in different curricular areas and to support children in applying their mathematical knowledge in contexts beyond the mathematics classroom. Here, we consider how proposals to integrate mathematics across the curriculum might impact on curriculum development and implementation, with reference to similar efforts in other jurisdictions.

In discussing the use of mathematics across of range of contexts in early childhood, Perry and Dockett (2002) note that 'the development of mathematical knowledge and skill go hand-in-hand with their application. Just as mathematics is learned in context, so it is used in context to achieve some worthwhile purpose' (p. 82). This view is consistent with the 'integrate and connect' process skill in the current PSMC (Government of Ireland, 1999a), which includes:

- connecting informally-acquired mathematical ideas with formal mathematical ideas
- recognising mathematics in the environment
- carrying out mathematical activities that involve other areas of the curriculum.

Perry and Dockett (2008) note that 'the contextual learning and integrated curriculum apparent in many early childhood – particularly prior-to-school – settings ensures that there is little distinction to be drawn between numeracy, mathematical literacy and aspects of mathematical connections with children's real worlds' (p. 83).

In Ireland, the *National Strategy to Improve Literacy and Numeracy 2011–2020* (DES, 2011a) emphasises the value of teaching numeracy across the curriculum. The cross-curricular emphasis in the strategy arises from (a) a recognition of the importance of numeracy; and (b) the need to ensure that children have additional opportunities to engage in mathematics, beyond dedicated mathematics time.

In recent years, a number of jurisdictions have begun to emphasise the application of mathematics across the curriculum and across the school day. In Australia, the Australian Curriculum, Assessment and Reporting Authority (ACARA) argues that 'using mathematics skills across the curriculum both enriches the study of other learning areas, and contributes to the development of a broader and deeper understanding of numeracy' (n.d., p. 1). It states that, in order to promote numeracy, teachers need to

- identify the specific numeracy demands of their learning area
- provide learning experiences and opportunities that support the application of students' general mathematical knowledge and skills
- use the language of numeracy in their teaching as appropriate.

To support teachers in doing this, ACARA provides six numeracy learning levels covering foundation year to grade 10. For each level, examples of relevant applications are provided, and specific objectives in English, mathematics, science and history are cross-referenced with examples of cross-curricular mathematical activities.

Using Mathematics is identified as one of three key cross-curricular skills in Northern Ireland (the others are Communication and Using ICT). As well as teaching mathematics as a curriculum area in its own right, teachers are required to teach and encouraged to assess mathematics as a cross-curricular skill. At Key Stage 3, teachers are provided with levels of progress in the cross-curricular skills to support them in assessing children (though the assessments are not statutory). The levels, which are intended to span the primary school years, include reference to

- choosing the appropriate materials, equipment and mathematics to use in a practical situation
- using mathematical knowledge and concepts accurately/working systematically and checking their work
- using mathematics to solve problems and make decisions/developing methods and strategies, including mental mathematics
- identifying and collecting information/reading, interpreting, organising and presenting information in mathematical formats
- using mathematical understanding and language to ask and answer questions, talk about and discuss ideas and explain ways of working
- developing financial capability
- using ICT to solve problems/present their work (CCEA, n.d-b.).

In engaging in these activities, children are expected to draw on their knowledge and understanding of number, measures, shape and space and handling data. There is no doubt that this initiative represents a considerable challenge to teachers, both in terms of teaching and assessment, and it is unclear at this time what advantages accrue from making it a focus of assessment.

While the curriculum initiatives in Australia and Northern Ireland designed to promote numeracy across the curriculum are indeed innovative, there is limited evidence to support them, though, as noted in Report No. 17 and in *Chapter 5* in this report, some effects have been found for integrating literature and mathematics. If the redeveloped mathematics curriculum for 3- to 8-year-olds promotes children's use of numeracy across learning areas, by, for example, cross-referencing learning outcomes in mathematics with learning outcomes in other curricular areas, it would be important to consider the supports that teachers might need to implement the model, and to evaluate its effects on children's mathematical development. In particular, it would be important to examine how adept teachers are at identifying opportunities to integrate mathematics into other subject areas, and what supports they might need to do this, particularly in the early stages of integration. It would also be important to ascertain if the allocation of time to the integration of mathematics (as suggested by the *National Strategy to Improve Literacy and Numeracy*) is as effective in promoting children's mathematical understanding as allocating additional time to mathematics as a specific area of learning in its own right.

Conclusion

While the specification of processes and content in the curriculum is critically important, attention must also be given to other issues that support optimal implementation. Among these is access by all children, including exceptional children and children in culturally diverse contexts, to the mathematics curriculum. Other issues relate to the timing of early intervention in mathematics, the allocation of time to mathematics in early learning settings, and the integration of mathematics across the curriculum. If addressed appropriately, and in combination with good mathematics pedagogy, they will contribute towards the realisation of an equitable mathematics curriculum.

The key messages arising from this chapter are as follows:

- The curriculum should recognise the particular challenges of working with exceptional children (including those with intellectual and developmental difficulties, and children with mathematical talent) and with children in culturally diverse contexts.
- A range of approaches and interventions is needed to ensure that children in disadvantaged circumstances reach their full potential in mathematics learning.
- While good mathematics pedagogy meets the needs of all children, an emphasis on mathematical discourse can play a significant role in supporting mathematical learning in the range of language contexts (e.g., English language learners and children in Irish-medium schools).
- A much earlier entry point than is indicated in the *Learning Support Guidelines* is suggested for the provision of early intervention designed to meet the needs of children who are at-risk of experiencing mathematical difficulties. Support for these children should focus on modifications to pedagogy designed to address their needs.
- Sustained time in preschool and primary school contexts – both dedicated and integrated – is an important precondition for children's engagement in mathematics learning.
- It is important that children engage with all of the domains of mathematics, and that opportunities to establish connections between domains are maximised.
- The mathematics curriculum should support an integrative approach across learning areas. One of the ways that this might be achieved is by the provision of exemplars illustrating good practices.

CHAPTER 5

Partnership with Parents

The important role of parents in supporting their children's mathematical development was referred to several times in Report No. 17, in the context of describing approaches to developing mathematical understanding (see Chapter 3, Section: *Adult Support*) and assessing mathematics (see Chapter 6, Section: *Supporting Children's Progression with Formative Assessment*). In this chapter, we take an in-depth look at how parents can support their children's mathematical development, in partnership with preschools/schools and the wider community. First, we consider how parents can support children's mathematical development in the context of the broader relationship between home and educational settings. Second, we examine effective programmes and partnerships for enhancing children's mathematical learning, and stress the importance of a two-way flow of information. Third, we focus on the role of discussion between parents and children in developing mathematical concepts and mathematical language. Fourth, we look in more detail at initiatives involving parents and teachers, including those implemented in disadvantaged contexts and those on reporting to parents. Fifth, we look at specific activities at home in which parents can engage with their children to develop mathematical understanding. In describing these activities, we emphasise the importance of children's agency in managing their learning.

Parents and Their Children's Mathematical Learning

Epstein (1995) promotes the idea that parental and community involvement should be encouraged to acknowledge the major spheres of influence that affect children's learning: family, school and community. It is generally accepted that the home-learning environment has a powerful effect on children's educational achievement (Anthony & Walshaw, 2007; Epstein, 1995; LeFevre et al., 2009). One of the targets listed in the *National Strategy to Improve Literacy and Numeracy* (DES, 2011a) is to enable parents and communities to support children's numeracy development. However, although decades of research have been conducted on home literacy experiences and strong recommendations for certain parental practices have been made, the same level of research has not been carried out into children's early mathematical experiences (LeFevre et al., 2009, p. 55). Anthony and Walshaw describe the situation as follows: 'Parents know what it means to *read* with children, yet they are often unclear about what it means to *do mathematics*' (2009b, p. 161, original italics). Maher (2007) found that partnerships between parents and teachers of young children were not as established in mathematics as in reading in a New Zealand-based study. It is likely that this may also be the case in Ireland. Literacy initiatives that target the wider community also appear to be more common than community mathematics initiatives. The Home School Community Liaison (HSCL) Coordinators (2006) describe a number of literacy initiatives where home, school and community work together. One such initiative is the 'One book, one community' scheme, where all members of the community are encouraged to read and discuss the same book (O'Brien Press, 2013). Examples of similar community-based initiatives targeting mathematics are harder to find.

When working with parents to support their children's mathematical learning, it may be necessary to address parents' lack of confidence in their own understandings of mathematics and/or possible alienation (Muir, 2012). Cannon and Ginsburg (2008) note gaps between parents' attitudes to and understandings of mathematics in the early years and highlight the need to educate parents about the mathematics children may learn through daily activities. Changes in approaches to the teaching of mathematics may also make it difficult to involve parents. There is a need to educate parents on the purposes and nature of current approaches, as they might not value such aspects as the incorporation of games or manipulatives (Civil, 2006; Muir, 2012). It is likely that the same holds true for any proposed curriculum change. Bleach (2010) discusses the possibility of 'overload' in Irish primary schools in relation to policy changes and states that it is difficult for schools both to implement changes, and inform and educate parents about those changes. However, she notes that parents cannot effectively support their children if they do not understand the changes being made.

There is a danger that teacher-led practices aimed at involving parents, particularly those in lower socio-economic groups, in their children's education might be based on a deficit model where is envisaged that input is needed from educators to correct a perceived deficit in the home environment (Edwards & Warin, 1999; Hanafin & Lynch, 2002; Whalley, 2001). However, the literature suggests that, in general, parents of all socio-economic backgrounds wish to support the mathematics education of their children (Anthony & Walshaw, 2007). Parental involvement often involves mothers rather than fathers (Byrne & Smyth, 2010; Whalley, 2001, 2007), and parents in higher socio-economic groups are often more visible in formal involvement with schools as members of parents' associations or boards of management (Bleach, 2010; Hanafin & Lynch, 2002). However, this involvement frequently has limitations and Hanafin and Lynch (2002) report parents' feelings of frustration with these formal structures, where their input is limited to fundraising and 'rubber-stamping' school initiatives rather than shaping them.

At times, there appears to be a mismatch between Irish primary parents' and teachers' perceptions of desirable parental involvement in formal curriculum matters (Mac Giolla Phádraig, 2003a). MacGiolla Phádraig (2003b) notes that, although neither parents nor teachers appear to want to increase levels of parental involvement at policy level, this stance is at variance with official departmental policy. Disparate opinions on desirable levels of parental involvement in their children's mathematics education may also be linked to value judgements about whose knowledge about mathematics and education is valid, with a higher value often being put on the knowledge of professionals by both parents and teachers alike (Merretens, 1993; Conaty, 2006a).

It seems that partnership approaches should inform the earliest stages of planning interventions and should involve key stakeholders – parents and teachers/practitioners – from the outset (e.g., Whalley, 2001, 2007). From their review of research in this area, Desforges and Abouchaar (2003) suggest that in addition to goodwill and a willingness to work, the following conditions are necessary to increase effective parental involvement:

- strategic planning which embeds parental involvement schemes in whole-school development plans
- sustained support, resourcing and training
- community involvement at all levels of management from initial needs analysis through to monitoring, evaluation and review
- a commitment to a continuous system of evidence-based development and review
- a supportive networked system that promotes objectivity and shared experiences (p. 70).

They also maintain that positive effects on student achievement will result from attention to specific educational goals. In the case of young children learning mathematics, this would suggest that parental involvement activities should target specific mathematical learning goals. The *National Strategy to Improve Literacy and Numeracy* also suggests that engagement with parents should be a core part of schools' literacy and numeracy plans (DES, 2011a), and this is further emphasised in the literature that supports school self-evaluation (DES, 2012).

Communicating with Parents about Mathematics

It is sometimes suggested that a 'communications gap' exists between teachers and parents where teachers' professionalism may act as a barrier to genuine communication (Crozier, 2000). Parents of children attending schools in economically disadvantaged areas have reported feelings of unease when talking with teachers (Hanafin & Lynch, 2002). Parents in socio-economically disadvantaged areas have also been found to be more reluctant to question the teacher or ask for clarification (Hall et al., 2008). Communication issues can be compounded when parents have literacy problems, or additional language issues (Evangelou, Sylva, Edwards, & Smith, 2008) and would appear to be particularly prevalent in the discussion of mathematics, which seems to be less accessible for some parents than other subject areas (Merttens & Newland, 1996). Effective programmes and partnerships often take a holistic approach where sustained mutual collaboration leads to the development of long-term relationships that support positive social change (Anthony & Walshaw, 2009b; Bleach, 2010; Evangelou et al, 2008; Goos, Lowry, & Jolly, 2007; Galvin, Higgins, & Mahony, 2009). This understanding of partnership foregrounds a two-way flow of information.

Sharing Information with Parents

The guidelines for working with Parents outlined in *Aistear* (NCCA, 2009b) recommend the sharing of information with Parents about the curriculum, about children's progress and their learning activities. The guidelines suggest ways of communicating with parents about mathematics, for example, formal meetings about the nature of the curriculum and methodologies, using notice boards, newsletters or photographs to document activities, and providing suggested activities for home. It is also recommended that resources could be shared with parents and that parents could be invited to spend time in the setting. The teacher guidelines accompanying the PSMC also recommend a whole-school sharing of information with parents, particularly in regard to reports of children's progress, and whole-school policies in mathematics and homework (Government of Ireland, 1999b). This contrasts with the less formal approach which has been reported in some early childhood care and education settings. In a study of community childcare centres in the Dublin Docklands, which used the Pen Green approach to parental involvement (Whalley, 2001), 'a quick chat' was often the expected level of parental involvement, for both parents and practitioners (Share, Kerrins, & Greene, 2011, p. 7).

A key practice of the Pen Green Centre is the enhancement of parents' understandings of key child development concepts, sometimes using video-clips of children's activity, so that parents may recognise schemas or patterns of behaviour used by their children (Arnold, 2001). This practice acknowledges the role of parents both as educators and learners. Parents of children attending the Pen Green Centre are offered the opportunity to make recordings of their children engaged in activities at home and discuss it with the centre staff. This could be considered a step forward from transmission-model workshops where curriculum information is simply presented to parents. Instead, this model expects parents to act as educators too and values their observations about their child's activity. Finding ways to support parental understandings of key stages of development in mathematics would go some way towards helping parents support the mathematical development of their children. Parents may benefit from discussions about what constitutes mathematically-rich activity for young children (see Chapter 2, Section: *Practices in Integrative Contexts*).

A Two-way Flow of Information

Efforts to create a two-way flow of information appeared hard to implement in practice when the Pen Green approach was used in community childcare centres in Dublin's Docklands (Share et al., 2011). This scheme recommended keeping a portfolio of children's work as a starting point for discussion with parents. Epstein (1995) also suggests sending home a portfolio of children's work on a weekly or monthly basis for comments and review as one means of attempting to facilitate two-way communication. Traditional ways of communicating with parents include the use of parent-teacher meetings and reports of student progress. Mac Giolla Phádraig (2005) suggests that parent-teacher meetings should go beyond a one-way flow of information where teachers provide a report that is generally based on summative assessments of learning (Hall et al., 2008). Instead, these meetings provide an opportunity to agree on priorities for the child's education and for the sharing of information (Mac Giolla Phádraig, 2005). Anthony and Walshaw (2007) suggest that parents may contribute to assessment and Shiel, Cregan, McGough and Archer (2012) note the possibilities for parents to contribute information on their young child's oral language in settings beyond the school or Early Childhood Care and Education (ECCE) setting. Some teachers reported that the mid-year report templates, which were used in an NCCA school-based development initiative, supported effective communication at parent-teacher meetings (NCCA, 2008). Some report templates included the sections 'ways you can help your child' or 'next steps in your child's learning' for explicit advice to parents on how to support their child's learning. Parents reacted positively to the inclusion of these pointers. However, some teachers found it challenging to complete the templates in this manner and were reluctant to give formal advice that might create pressure for parents and children. Since 2011, all schools are required to use NCCA templates to report to parents (DES, 2011b).

The HSCL Scheme is an example of a scheme which aims to increase cooperation in the education of students between schools, parents and community agencies as a means of addressing disadvantage (Ryan, 1994). The goals and principles of the scheme are outlined by Conaty (2006b), who notes the positive possibilities of partnership in terms of its potential for empowerment of individuals and transformation of relationships. A two-way flow of information is envisaged in the 'local committee' element of the HSCL scheme which was described above. At primary level, there is some involvement in classroom activities such as paired reading and targeted programmes such as *Mathematics for Fun* (HSCL Coordinators, 2006). Almost all principals and coordinators reported a positive impact on students, parents and schools (Archer & Shortt, 2003). However, in general, it was perceived that the parents most in need of assistance did not become involved in HSCL projects.

Humphrey and Squires (2011) report on a major intervention, *Achievement for All* (AfA) across 454 schools in England designed to support schools and local authorities to provide better opportunities for learners with special educational needs and disabilities (SEND) to fulfil their potential. AfA had a significant impact on progress in mathematics and English. A key component identified in the success was the structured conversations with parents. These focused on the use of a clear framework for developing an open, on going dialogue with parents about their child's learning. Training was provided for schools, which emphasised the building of parental engagement and confidence via a four-stage model (explore, focus, plan, review) in up to three structured conversations each year with parents in reviewing individual goals.

Technology and Communicating with Parents

Developments in information technology can also create opportunities for new ways of communicating (Desforges & Abouchaar, 2003). In fact, the Ballymun Whitehall Area Partnership with Hibernia Consulting (2009) interviewed a number of parents and found that, apart from people they meet or speak to, the internet was the main source of information for how to support their children's learning. Preschools and schools can make more use of technology to communicate with parents. For example, schools and ECCE settings could build on the 'tip sheets' and videos of sample activities for parents available on the NCCA website to provide details on appropriate mathematical learning activities for different age levels. School and ECCE websites could provide information about, and examples of, appropriate mathematical learning activities, resources and links to further information. They could also use websites to detail (with videos, digital photos, samples of student work etc.) the on-going mathematical learning activities of the ECCE setting/classroom and suggest follow-up activities for parents. Also ways to facilitate feedback from parents, either in a face-to-face or online, could be considered. It is possible that parents themselves may use tools such as video or digital cameras to record their child's mathematical activities in the home. Clarke and Robbins (2004) report on a study where parents were provided with a disposable camera and asked to document their preschool children's literacy and mathematical activity. It seems that the task itself increased awareness of the range of mathematical experiences that occur in daily life: parents categorised photos as related to number and other aspects of mathematics. The range of mathematical activities appeared to surprise teachers, and it is likely that variations on this project may be useful for developing discussions between teachers and parents as well as between parents and children.

Parents and Children Discuss Mathematics

Benigno and Ellis (2004) suggest that differences in young children's mathematical abilities may be related to the different kinds of social activities in which they engage in at home. Research suggests that although parents engage in a variety of numerical activities with their children, they do not always utilise opportunities to promote their child's numeracy skills (Benigno & Ellis, 2004; Tudge & Doucet, 2004; Vandermaas-Peeler et al., 2012). The nature of the content of mathematical discussion between parent and child is important and effective parent-child mathematical discussion should move beyond counting to incorporate more complex goals (Skwarchuk, 2008). Skwarchuk found that, while parents often focused on number sense, they were unsure how to incorporate other areas such as measure.

The significance of discussion between parent and child as an influence on student achievement is supported by Desforges & Abouchaar (2003). Sheldon & Espstein (2005) suggest that activities where families engage in discussion on mathematics while engaging in mathematics activities may contribute to improving children's mathematical skills. This is further supported by Siraj-Blatchford et al. (2002), who report a positive effect on young children's achievement when parents used discussion-based learning activities at home. As reported earlier, discussion-based learning is more effective when characterised by sustained shared interactions (see Chapter 2, Sections: *Promotion of Math Talk* and *Interactions during Story/Picture-Book Reading*).

The nature of parent-child interaction is important, with Pomerantz, Moorman, and Litwack (2007) contending that effective parental interaction supports the development of the child's autonomy, is focused on the process of learning rather than the performance of the child, and is characterised by positive parental affect and positive beliefs about the child's potential. This has saliencies with the descriptions of two contrasting parental pedagogical styles described by Aubrey, Bottle and Godfrey (2003), one didactical and one where the focus was on the child's participation. One mother appeared to recognise communication as important, viewed mathematics as part of everyday life and viewed learning opportunities as likely to arise from play. In contrast, another mother took a more direct teaching role and appeared to see mathematics in the home as a series of discrete activities with counting and arithmetical skills as a primary goal. The authors note the potential for a possible disconnect with school, where teaching approaches may sometimes be perceived by parents as more formal than those in ECCE settings. Vandermaas-Peeler, Nelson, Bumpass, and Sassine (2009) recommend that parents should be provided with suitable examples of mathematical discussion that may arise in day-to-day life and that early childhood educators should encourage parents to incorporate mathematics into readings of picture books as well as play activities.

Parents and Teachers Collaborating about Mathematics Learning

There is a number of possibilities for parental involvement in schools, including the use of parents' rooms, curriculum meetings or workshops, and parents assisting on outings or in the classroom (Border & Merttens, 1993). Some of these activities are also recommended in the *Aistear* framework (NCCA, 2009b), in the Home School Liaison Scheme (HSCL Coordinators, 2006) and in various other policy documents (e.g., DES, 1995). Having parents physically present in schools makes visible efforts to encourage parental involvement. However, simply 'getting the parents in' does not guarantee effective practices (Edwards & Warin, 1999; Sheldon & Epstein, 2005). Authors note that subject to how activities are enacted, parents are generally still positioned outside the locus of control of curriculum and pedagogy (Border & Merttens, 1993; Hanafin & Lynch, 2002; Hallgarten, 2000; Mac Giolla Phádraig, 2005), although Anthony and Walshaw (2009b) discuss a number of effective schemes where parents and teachers engaged in genuine collaboration to develop teaching and learning activities.

Muir (2012) describes a 'maths club' which was set up in response to a perceived need to support the parents of students in the senior end of primary school. Attendance rates varied but engagement by parent participants was enthusiastic. The maths club attempted to challenge traditional attitudes to mathematics and activities were based on identified areas of need. The activities gave parents an opportunity to explore how mathematics topics are approached in the contemporary classroom and provided an opportunity for reviewing preconceptions. Muir notes that the workshops not only address mathematical content but also served a purpose in the affective domain by increasing the confidence of parents and their motivation to do mathematics with their children. This may be particularly important for parents who did not have a positive experience of learning mathematics themselves. It is likely that Muir's approach could be adapted for younger age groups. Civil, Quintos and Bernier (2003) implemented a slightly different approach in the US. While both parents and children were involved in workshops, the children were dismissed during the later stages of the workshops to allow parents to function as adult learners and discuss children's thinking. Such workshops have the possibility to be effective, particularly if parents have some input into the mathematics topics chosen. In both studies, initial workshops were led by outside researchers and the second study used initial outsider input to train local facilitators. This underlines the need for adequate planning and resourcing of parental involvement projects.

Civil et al. (2003) also arranged parent observations of mathematics classes as a means of facilitating dialogue with parents. The dialogue that followed the observations highlighted how parents' own experiences of learning mathematics shaped their perspective of mathematics lessons and what they value as important or 'good' practice. This study focussed on dialogue between researchers and parents rather than parents and teachers. However, it might be possible to adapt the approach to facilitate parent-teacher or parent-practitioner dialogue, possibly using video-recordings for those parents unable to attend during working hours. Such recordings or observations would make it possible for parents to experience approaches to the teaching of mathematics that may be quite different from how they learned the subject (Civil et al., 2003). Having a section of a school or ECCE setting's website dedicated to such video clips could serve as a point of discussion between both parent and child and parent and teacher. Such clips might also provide support to parents and children when engaging follow-up mathematical activities or homework.

It is suggested that HSCL coordinators may function as the 'driving force' behind literacy and numeracy initiatives in schools (HSCL Coordinators, 2006, p. 147) and, as such, it is possible that they may play a role in any new initiatives targeting numeracy activities with parents of children attending DEIS schools. *Mathematics for Fun* is a programme run under the HSCL scheme where parents are invited to participate in and support children's mathematical activities in schools (HSCL Coordinators, 2006). The mathematical activities often focus on the use of manipulatives including tangrams, pattern blocks, dominoes and pentominoes (ibid.). Parents are invited to attend training sessions in the use of the mathematical activities or games and sessions are held over a six-week period, with each session lasting roughly one hour a week. It is suggested that class teachers should be consulted about the suitability of materials but it is unclear how much input the class teacher has into the choice of mathematical activities and it is hard to judge how closely related these activities are to the class scheme for mathematics. Positive feedback from parents, teachers and children about the nature of engagement in activities is reported (HSCL Coordinators, 2006). An evaluation of educational partnerships between Mary Immaculate College, five primary schools, parents, community groups and other organisations lists a wide range of creative partnership projects.

However, numeracy was generally only targeted through the *Mathematics for Fun* programme, related science-focused programmes or through the creation of chess clubs (Galvin, Higgins & Mahony, 2009).

It seems likely that support and training are needed to develop and extend the use of mathematics-focused parental involvement programmes and to broaden the extent of such partnerships to include other members of the community. It also seems likely that such programmes may more directly impact student achievement if they are closely tied to the class learning plan (Desforges & Abouchaar, 2003).

Parent and Child Collaborating about Mathematics

There are a number of mathematical activities that can be used to support young children's learning at home and at school. These include digital and traditional games, number and shape books, number songs and other activities that make use of the environment such as discussions of calendars or money (Vandermaas-Peeler et al., 2012). Anderson, Anderson and Shapiro (2005) examined parent-child exchanges during shared reading sessions with 4-year-old children in their homes. Their findings suggest that, while there was considerable diversity with regard to the type of interactions and the manner in which parents engaged with their children, all but one integrated math talk into the story-reading, especially when discussing the illustrations. The story-related conversations in the home were not in any way contrived and the focus was on the co-construction of meaning. Particular attention centred on concepts such as size and number, as these were seen as arising in a meaningful way within the context of the story.

Homework can be used as a means of facilitating opportunities for parental participation in their children's learning (Merttens & Newland, 1996; Sheldon & Epstein, 2005). Interactive homework may include activities that require parent-child interaction about mathematics or the use of mathematical materials and resources that may be provided by the school (Sheldon & Epstein, 2005). The *Impact* project was a large scale project carried out in the UK which involved parents completing interactive mathematical activities at home with their children (Merttens & Newland, 1996). It aimed to increase opportunities for a two-way flow of information by including opportunities for feedback from parents. Feedback included comments on how enjoyable and accessible the activities were as well as providing opportunities for parents to informally assess their children's mathematical learning. Merttens and Newland (1996) note that having parents assess how their children engaged in the task facilitated more in-depth discussion at later parent-teacher meetings, transforming the experience from 'teacher monologue' to 'dialogue'. They note also that negotiating approaches to a 'school-mathematics' task in the home allows for the regulation and articulation of the task to move between parent and child, particularly when the child acts as instructor to a 'task-naïve' adult (p. 111). This may allow the parent to interact with the child in a way that supports his/her autonomy. This is generally believed to be an effective form of interaction (Pomerantz et al., 2007). Factors affecting the uptake of *Impact* activities included the manner in which they were introduced to parents, the expectations that were set around acceptable levels of parental involvement and the teacher and his/her role in maintaining contact. The completion of specific activities was sometimes linked to whether parents viewed the activities as 'maths' and whether it related to their understandings of what mathematics education should consist of. This echoes the observations of other authors on parents' perceptions of newer approaches to mathematics (Civil, 2006; Muir, 2012; Pritchard, 2004).

Muir's research on family involvement with 'take home mathematics packs' (2009, 2012) was also based on a similar approach. Muir (2012) describes how activities were designed or selected by the teacher and researcher, based on links to classroom activities, and how a number of 'numeracy bags' were prepared. These bags contained instructions for the activity, necessary materials and guidelines for parents as well as a rationale detailing the mathematical purpose behind the activity. It was intended that children engage in these activities at home with their families two to three times over the course of a week before exchanging the pack for a new numeracy bag. Each numeracy bag also contained a feedback sheet for parents to comment on their child's engagement and any mathematical understandings that were noted. This feedback goes some way to developing communication between parent and teacher.

A strength of both the *Impact* scheme and Muir's scheme is that both focus on numeracy activities that were closely tied to the class learning goals (Desforges & Abrouchaar, 2003). For any practitioner or teacher attempting to initiate such a scheme, attention should be paid to observations of Pomerantz et al. (2007) discussed above. Tasks should be designed and presented in such a way that parents do not feel under pressure to ensure that their children perform in particular manner as this may hinder their inclination to act in process-focused ways that support the autonomy of their children (Pomerantz et al., 2007). Any initiative should be designed with the aim of developing and maintaining the positive affect of parents and positive beliefs about the potential of their children (ibid.)

Conclusion

Parents can become involved in their child's mathematics learning in a variety of ways. This involvement can have positive effects on children's learning. Parental involvement in early education settings should be characterised by a two-way flow of information. Early years educators should highlight with parents the importance of engaging in discussion with their child about mathematically-related activities that arise in the home, and in the context of homework when appropriate. Parents and teachers collaborating and sharing information has been found to be advantageous to teachers, parents and children.

The key messages arising from chapter are as follows:

- There is a need to inform parents about the importance of mathematics learning in the early years, and what constitutes mathematical activity and learning for young children.

- Communication with parents needs to emphasise how the redeveloped curriculum can foster children's engagement with mathematics and the significant role of parents in supporting children's learning.
- Digital technologies offer great potential for communication between parents and educators about young children's mathematical development.
- There is a range of activities in which parents can engage with schools so that both parents and educators better understand children's mathematics learning.

CHAPTER 6

Teacher Preparation and Development

As already elaborated in this and the previous volume, the shift in perspective on what it means for children to learn and use mathematics in the early years demands a change in pedagogy; in particular it puts the teaching-learning relationship at the heart of mathematics (Report No. 18, Chapter 1). This shift requires that educators engage in mathematics teaching in a manner that is qualitatively different from how they themselves learned mathematics (Corcoran, 2008). Putting the teaching-learning relationship at the heart of mathematics acknowledges the equable, or otherwise, outcomes of particular teaching practices and the influence of teacher beliefs and attitudes on these (Bibby, 2009). In the context of defining pedagogy as being about relationship, Bibby (2009, p. 123) contends that '[r]elationships are hard work: They involve knowledge and thinking that goes beyond the rational'. She frames them in terms of 'being in and with' (p. 127) the efforts of learners in the mathematics classroom. This requires that educators be personally committed to teaching mathematics well.

If children are to learn and use mathematics in the coherent and connected manner outlined in Volumes 1 and 2 of this report, educators must acknowledge its importance as more than 'just one of the subjects that I have to teach' (Brown & McNamara, 2005, p. 11). This shift in perspective from traditional teaching of mathematics as rules and procedures, to mathematics teaching as developing mathematical proficiency has important implications for teacher education and sustained teacher development. Among the implications for teacher preparation and development that we have already highlighted as important are:

- a background knowledge of developmental progressions in mathematics learning
- provision for a diverse range of learners
- familiarity with the principles and features of good mathematics pedagogy
- understanding of the role of play in young children's mathematics learning
- incorporation of key meta-practices (math talk, the development of a productive disposition, an emphasis on mathematical modeling, the use of cognitively challenging tasks and formative assessment) in everyday mathematics activity

- a focus on the overall aim of mathematical proficiency and cognisance of the key role that mathematisation plays in progressing this aim.

It is recognised in the literature that teacher preparation and development are complex endeavours that benefit from community support (Krainer, 2005; Jaworski, 2006; DES, 2011a). Below, we discuss key ideas arising from research that should inform professional development.

The Goal of Mathematics Teacher Preparation

We have identified mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) as a key aim of mathematics education. In terms of early years teaching, teachers need a strong working knowledge of mathematics and an openness to and facility for problem-solving. Much of the research in mathematics teacher education of the past twenty-five years has focused on 'a well organised and flexibly accessible domain-specific knowledge base' (De Corte, 2004, p. 282). Changes in perspective on what it means to know and use mathematics in teaching can be seen where the research emphasis has moved from the importance of teachers' subject matter knowledge of mathematics (Ball, 1988) to a widespread acceptance that pedagogical knowledge is the more important teacher variable in student achievement (Education Committee of the European Mathematical Society (EMS), 2012). It follows that the emphasis in mathematics teacher preparation programmes must rest equally on developing both mathematics and mathematics education. The two goals of mathematics teacher preparation must therefore be a) to inculcate a mathematical disposition in future educators, b) together with learning the pedagogic skills and competencies to foster and promote mathematical proficiency in their children (Hiebert, Morris, & Glass, 2003). The design and provision of mathematical experiences that lead to progressive development of each of the strands of mathematical proficiency is primarily the responsibility of the educator in the learning environment. It requires a pedagogy underpinned by principles which relate to people and relationships, the learning environment and the learner (see Chapter 1, Section: *Features of Good Mathematics Pedagogy*). To do this effectively, educators need substantial knowledge of mathematics. They need to develop skills for promoting math talk, developing productive dispositions, emphasising mathematical modeling, selecting cognitively challenging tasks and assessing learning (see Chapter 2, Section: *Meta-Practices*).

Mathematical Knowledge for Teaching (MKT)

The construct of MKT is claimed to conceptualise the specialised knowledge teachers need in order to complete the tasks of teaching mathematics (Ball & Bass, 2003). MKT is thought to consist of Subject Matter Knowledge and Pedagogical Content Knowledge. Subject Matter Knowledge is further categorised as 'common content knowledge' (CCK) and 'specialised content knowledge' (SCK). CCK has been identified by recent research in the US as mathematical knowledge present in the population at large (Ball, Thames, & Phelps, 2008) and SCK represents the 'specialised' knowledge of mathematics teachers need in order to teach mathematics successfully. Ball and colleagues adapted Shulman's (1986, p. 9) pedagogical content knowledge (PCK) 'for teaching [mathematics]' as Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC), all of which they claim can be measured psychometrically.

MKT Research in Ireland

Elements of mathematical knowledge for teaching (MKT) have been enumerated as responding to children's questions, choosing useful examples for highlighting salient mathematical issues, planning lessons, appraising and modifying textbooks and assessing children's learning (Delaney, 2010). These 'tasks' arise out of research of teaching mathematics in the US and Delaney has drawn attention to the mathematical work of teaching in Ireland and the mathematics Irish teachers 'know'. He adapted multiple-choice items developed by Ball, Bass and colleagues for use in Ireland. In these items, teachers are asked, for example, to identify child errors (from a classroom scenario) or to select appropriate representations for particular mathematics problems. The main finding of his study is that mathematical knowledge for teaching varies widely. While his work is based primarily on records of existing practice, Lampert (2001) and Boaler and Staples (2008) propose a relational approach to the development of teacher knowledge – that is, where the educator empathises with the learners as they come to know mathematics. This is discussed further below (see *Effective Teachers' Framework*).

Profound Understanding of Fundamental Mathematics

A small but highly respected study by Ma (1999) demonstrates how powerful mathematics teacher knowledge can be. She offers a comparison of the performance of American and Chinese teachers on four mathematical domains and finds that Chinese teachers demonstrated a consistently stronger conceptual understanding of mathematics than the American teachers and were in all instances better able to explain their 'knowledge packages'. This was despite them having less formal training for teaching mathematics. From her research, Ma describes what she considers essential for teaching: a profound understanding of fundamental mathematics (PUFM). It represents a form of mathematical knowledge that is highly connected, and strongly geared to teaching. In her concluding chapter, Ma argues that what American teachers need is not 'more mathematics' but a 'refocus on teacher preparation' which involves:

> *Rebuilding a solid and substantial school mathematics for teachers and students to learn… a substantial school mathematics with a more comprehensive understanding of the relationship between fundamental mathematics and new advanced branches of the discipline…indeed, unless such a school mathematics is developed, the mutual reinforcement of low-level content and teaching will not be undone.* (p. 149)

From a teacher preparation perspective, it is also worth paying attention to the collaborative and reflective practices among Chinese teachers, which undeniably influenced the development of PUFM. Undoubtedly, educators wishing to progress the aim of mathematical proficiency must have deep and connected knowledge of fundamental mathematics. We also know that effective teaching of mathematics requires considerably more of teachers than being able 'to do' the mathematics they plan to teach; for example, the teaching of subtraction to young children requires more of the teacher than merely knowing how to perform the standard decomposition algorithm (Rowland, 2007). In her research with pre-service teachers, Corcoran (2008) used an audit developed by a team of teacher educators for the SKIMA (Subject Knowledge in Mathematics) project in the UK (Rowland, Martyn, Barber, & Heal, 2001). This audit gives an indication of some of the mathematical strengths and weaknesses of student teachers. She found that while there was a close relationship between mathematics results on a SKIMA audit and mathematics results on the Leaving Certificate Examination, neither was indicative of the quality of mathematics lessons that the student teachers in her study conducted. This indicates that mathematics teaching is a highly complex activity and mathematics teacher preparation requires acknowledgement of the situated, social and distributed nature of mathematics teacher knowledge (Lave, 1988). Mathematics pedagogy then, like mathematical proficiency, is seen as a complex whole, a set of interconnected parts.

There is currently no common mathematics syllabus for pre-service early years or primary teachers. The PSMC and its accompanying *Teacher Guidelines* (Government of Ireland, 1999a; 1999b) indicate the mathematics that primary teachers need to teach and might be interpreted also as indicative of the mathematics that teachers need to know. In fact, teachers of primary mathematics for children aged 3–8 years need to be able to use substantive mathematics, including algebraic thinking, generalisations, equations, functions and graphs, mathematical reasoning and proof, if they are to challenge children to think mathematically (e.g., Ma, 1999).

'Doing' Mathematics

In order to ensure that all children have access to rich mathematics and powerful mathematical ideas, pre-service educators (and practising teachers) need to engage regularly in challenging mathematical activities. By doing interesting and appropriate mathematical investigations and tasks in groups, educators can learn about learning mathematics through 'mathematization' and the opportunities it allows for processes such as communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting. Effective teaching of mathematics requires facility with these processes. They can be developed and their implications for classroom practice can be maximised through explicit mathematical communication and regular, reflective communal engagement with rich mathematics tasks in various contexts. A problem may exist where educators have had previous negative relationships with mathematics and so teacher preparation programmes must respect and model effective mathematics pedagogy by:

a) acknowledging that all students irrespective of age or previous experience can develop positive mathematical identities and become powerful mathematical learners and teachers

b) responding to the multiplicity of thinking process and realities found in everyday classrooms with interpersonal respect and sensitivity

c) focusing on optimising a range of desirable academic outcomes that include mathematical proficiency and a mathematical disposition

d) committing to enhancing a range of social outcomes within the mathematics classroom that will contribute to the holistic development of participants for productive teaching (adapted from Anthony & Walshaw, 2009b).

By framing mathematics education courses along these lines, teacher educators can provide mathematically-rich learning environments that allow for personal agency and communal support so that pre-service educators may craft positive identities both as learners and teachers of mathematics. By addressing the development of a flexible knowledge base in fundamental mathematics, and engaging thoughtfully in appropriate problem-solving activities, teacher educators can also help their students to become aware of their self-regulation in learning and using mathematics.

Frameworks for Thinking about Pedagogy

The vision of 'good mathematics – taught well' (Even & Lappan, 1994) is integral to the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989, 1991, 2000). This is similar to the vision we propose for the redeveloped mathematics curriculum. Both the NRC report (2005) and Anthony and Walshaw's research synthesis (2007) emphasise the importance of frameworks, or systems for thinking about teaching, learning and the design of learning environments. Whichever mathematics curriculum one is teaching, much the same substantive and syntactic mathematics knowledge are required. So what is 'good mathematics' for Irish early years classrooms? And what constitutes 'good mathematics teaching' in Irish terms? The answer may lie in how the curriculum is interpreted and also in how mathematics is understood and valued (Huckstep, 2007; Corcoran, 2008). It is also based on a grounded understanding of learning pathways in mathematics.

The Knowledge Quartet

The Knowledge Quartet (KQ) is a powerful framework devised to aid the development of mathematical knowledge for teaching (Rowland, Huckstep, & Thwaites, 2005). Arising from studying primary mathematics teaching, the Knowledge Quartet identifies four dimensions along which teachers' mathematical knowledge impacts on teaching:

1. *Foundation* is knowledge of the mathematics to be taught, and of theories of teaching and learning mathematics. It includes attitudes and beliefs about mathematics knowledge and pedagogy.

2. *Transformation*, or knowledge-in-action, how to re-present ideas to make them better understood by children, resonates with Shulman's pedagogical content knowledge (1986). Transformation is manifest in a teacher's facility with the art of question-posing and the astute choice of examples.

3. *Connection* involves the ability to sequence material to be taught and an awareness of the relative difficulty for children of different curricular elements. It involves making connections between disparate parts of the lesson or series of lessons and resonates with the 'connectionist' teacher (Askew, 1999).

4. *Contingency*, the more generic ability to deal creatively with the unexpected direction in which a lesson may go, is arguably the most intellectually challenging and difficult to acquire of the four components of the Knowledge Quartet.

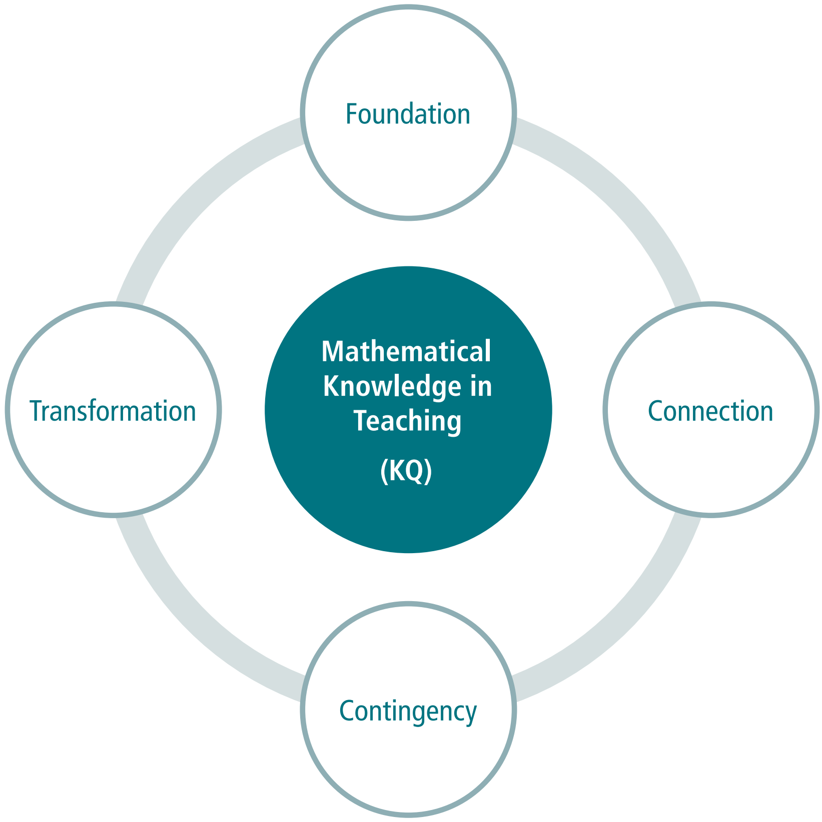

Figure 6.1: The four dimensions of the Knowledge Quartet feed into each other

Each of the four dimensions of the KQ is associated with a certain number of codes or indicators that can be identified with the teacher's activities in planning and teaching a mathematics lesson. The interconnectedness of the four dimensions is evident in the activity of teaching; however, Corcoran (2012) identifies contingency as the most central dimension and of greatest importance in developing mathematical proficiency. Contingency teaching is continually focusing on learners and responding to learners' ideas.

The KQ framework can be used in a number of ways to help in the preparation and delivery of mathematics teaching. It was devised originally to assist tutors who were not mathematics specialists in discussing lessons with student teachers. The twenty contributory codes that feed into the four dimensions of the Knowledge Quartet (Rowland et al., 2005) constitute a 'common technical vocabulary' for talking about teaching (Grossman & McDonald, 2008). Their use is recommended in teacher education as a means of talking about teaching, where they can help build community by becoming part of the 'shared repertoire of ways of doing things' (Wenger, 1998, pp. 82–84). They have been used successfully in a longitudinal study of primary mathematics teacher development in the UK (Turner & Rowland, 2011).

Table 6.1: Contributory codes to the Knowledge Quartet

KNOWLEDGE QUARTET DIMENSIONS	INDICATORS	
FOUNDATION	▪ Adherence to textbook ▪ Awareness of purpose ▪ Concentration on procedures ▪ Identifying errors	▪ Overt subject knowledge ▪ Theoretical underpinnings of pedagogy ▪ Use of terminology
TRANSFORMATION	▪ Choice of examples ▪ Choice of representation	▪ Teacher demonstration
CONNECTION	▪ Anticipation of complexity ▪ Decisions about sequencing ▪ Making connections between concepts	▪ Making connections between procedures ▪ Recognition of conceptual appropriateness
CONTINGENCY	▪ Deviation from agenda ▪ Responding to children's ideas ▪ Use of opportunities	▪ Teacher insight ▪ Responding to (un)availability of tools and resources

In the Knowledge Quartet framework, isolated practices are not the focus; rather it is the way in which the different elements of the system interact that is important.

Effective Teachers' Framework

Earlier we listed the main features of effective pedagogy by reference to the principles of people and relationships, the learning environment and the learner. Anthony and Walshaw (2009b) elaborate further on what effective teachers do in terms of pedagogy. They start with an 'ethic of care.' This principle underlines the relational aspects of mathematics teaching and learning and the educator's responsibility to be 'in and with' the learners as they 'struggle with mathematics' for themselves, as members of a mathematics learning community. This is followed by the responsibility of educators to 'arrange for learning' by putting learners' current knowledge and interests at the centre of their planning for play, individual, pair, group and whole class work as appropriate. Teachers who 'build on student's thinking' in the design of mathematical tasks are better able to adjust the complexity level of tasks to challenge low-achieving learners. Focus on children's thinking also helps teachers promote the processes of mathematization by knowing when and how to increase the task challenge level. The next two principles are related to classroom discourse. Effective teachers of mathematics facilitate 'mathematical communication' and model the use of appropriate 'mathematical language'. 'Assessment for learning' of mathematics is integral to classroom discourse and effective teachers provide learners with multiple pathways to evaluate and assess their own work. Effective teachers use a variety of 'worthwhile mathematical tasks' and help learners 'make connections' across mathematics, between different solution paths in problem-solving and between mathematics and everyday life. Effective teachers of mathematics carefully choose 'tools and representations' to stimulate and support learners' thinking. Finally, the 'knowledge and learning' of effective mathematics teachers is substantial and robust. It includes 'grounded understanding of students as learners'. In outlining their views of effective mathematics pedagogy, Anthony and Walshaw (2009b) have moved 'away from prescribing pedagogical practice, towards an understanding of pedagogical practice as occasioning students outcomes' (p. 158), with resultant implications for teacher preparation and development.

Using Tools for Teacher Preparation

As research perspectives on mathematics teacher education have moved from an emphasis on teacher knowledge to a more child-focused and community-based approach, elements of Shulman's proposed knowledge base for teaching have been revisited. His proposal was for a) propositional knowledge, b) case knowledge and c) strategic knowledge (Shulman, 1986). According to Shulman:

> *Case knowledge is knowledge of specific, well-documented, and richly described events. Whereas cases themselves are reports of events or sequences of events, the knowledge they represent is what makes them cases. The cases may be examples of specific instances of practice-detailed descriptions of how an instructional event occurred – complete with particulars of contexts, thoughts, and feelings.*

Silver et al. (2007) outline – among other means of learning to teach mathematics effectively–the modus operandi of the COMET (Cases of Mathematics Instruction to Enhance Teaching) project,

which used written case studies in mathematics to effect teacher development. The idea behind the project is that reflection on events in specific classrooms enables teachers to begin to think in a more general way about important matters in mathematics teaching and learning, such as, for example, the need for connections. Similarly, a digitally-based interactive teacher development programme comprising 'records of practice' has been devised between colleagues in the US and representatives of RME in The Netherlands (Fosnot et al., 2013). The re-imagining and re-configuring of the BEd programme in the Irish context constitutes an ideal opportunity to build four-year teacher preparation programmes that include a rich bank of 'cases' of early years mathematics teaching/learning in different mathematical domains. These cases could be based on mathematics learning in a range of education settings in Ireland. Borko, Jacobs, Eiteljor, and Pittman (2008) argue for the effectiveness of videos of children learning mathematics as a tool for teacher education. These new ways of teacher preparation and development emphasise collaboration.
They require pre-service teachers to gain experience with 'approximations of practice' and to focus on a core set of 'high-leverage practices' (Ball, Sleep, Boerst, & Bass, 2009; Grossman, Hammerness, & McDonald, 2009). Thus, strategic knowledge about learning paths in mathematics can be explored and developed before educators begin to teach in 'live' educational settings.

Mathematics Teacher Development (CPD)

International research on mathematics teacher development points to a strong need for mathematics teacher education that continues beyond teacher certification (Krainer, 2011). Teacher participation in professional development as teachers of mathematics has traditionally been very low in Ireland (Delaney, 2005). Except where a small number of teachers have pursued masters or diploma courses in mathematics education, or have been trained to deliver a specialised mathematics programme (e.g., learning support), a five-day summer course is the most likely form of professional development that teachers access. Among the recommendations for in-post teacher development is investment in stronger systems of clinical supervision across the preparation-induction boundary (Grossman, 2010). The notion of clinical supervision could mean an emphasis on developing good mathematics teaching practices through collaborative review and reflection on existing practice. This is important because inquiry as a stance has been advocated as a successful key to teacher change (Jaworski, 2006).

A meta-analysis of teacher professional development research in the US shows that growth in teaching can be achieved through

1. building teachers' mathematical knowledge and their capacity to use it in practice
2. building teachers' capacity to notice, analyse and respond to students' thinking
3. building teachers' productive habits of mind
4. building collegial relationships and structures that support collegial work (Doerr, Goldsmith, & Lewis, 2010).

From their findings it emerges that the key shift involved is one of agency for teachers: from programmes that try to change teachers to teachers as active learners shaping their own professional growth through reflective participation in professional development programmes and in practice. There are numerous examples of such programmes and practice. For example, in Report No. 17 (Chapter 6, Section: *Formative Assessment*) we discussed an Australian project which sought to improve mathematics and numeracy outcomes through working with developmental learning outcomes and a set of powerful mathematical ideas. Lesson study is a practice that is currently foregrounded in the literature as a significant development in school-based professional development.

In lesson study, publicly available records of practice or 'actionable artifacts' are important by-products (Lewis et al., 2006, p. 6). It offers opportunities at school and classroom level for enactment of critical inquiry into mathematics lessons (Jaworski, 2006). Noticing children's responses is an explicit objective and becomes integral to teaching through participation in lesson study. Reflection on mathematics teaching becomes public and is shared through a common language (e.g., that associated with Knowledge Quartet) for talking about teaching. This makes lesson study a particularly effective vehicle for mathematics teacher development (Krainer, 2011) and it is noteworthy that it is gaining support internationally (see, for example, Corcoran & Pepperell, 2011; Fernández, 2005; Hart, Alston, & Murata 2011; Peterson, 2005). Congruent with sociocultural theories of learning, already outlined in Report No. 17, teacher professional development has been found to be more effective when it is sustained, local and supported by the school community (e.g., Cochran-Smith, 2012; Morgan, Ludlow, Kitching, & O'Leary 2010). From this perspective, lesson study is particularly beneficial when enacted by a community of educators working in their own setting, that is, where colleagues are mutually engaged in the shared enterprise of developing mathematical proficiency in their learners. In designing in-service programmes in relation to the redeveloped mathematics curriculum for 3- to 8-year-old children, school-based lesson study should be given due attention.

Conclusion

Teaching-learning relationships are at the heart of mathematics learning. This requires educators to engage in mathematics teaching in a manner that is qualitatively different from how they themselves experienced the learning of mathematics. If pre-service and in-service educators of young children are to promote good mathematics learning, they must have a well-organised and flexibly accessible domain-specific knowledge base. They should have a strong working knowledge of mathematics and an openness to, and facility with, the processes of mathematization. The construct of mathematical knowledge for teaching is important for teacher preparation and development. It can be developed in different ways depending on the level of experience of the teacher. However, critical and collaborative inquiry needs to underpin all efforts to develop teachers' expertise.

The key messages arising from this chapter are as follows:

- In order to integrate key meta-practices in pedagogy (math talk, productive disposition, modeling, cognitively challenging tasks and assessment), teachers need a profound understanding of mathematics.
- A profound understanding of fundamental mathematics can be developed by educators through a collaborative focus on teaching and learning of mathematics.
- Educators can develop mathematical knowledge for teaching through engaging in rich mathematics tasks.
- The focus of pre-service and in-service teacher education programmes should be on children's engagement in mathematics and their responses to mathematical ideas – valuable contexts are case studies of children learning mathematics and the practice of lesson study.
- Reflective frameworks (e.g., Knowledge Quartet) facilitate critical inquiry and the use of a common language for talking about learning and teaching mathematics.

CHAPTER 7

Key Implications

The purpose of this report is to inform the redevelopment of the mathematics curriculum for children aged 3–8 years. It builds on the research presented in Report No. 17 (definitions, theories, stages of development, and progression). In addressing the issues on teaching and learning, we focused on research related to pedagogy and curriculum. We drew on a broad range of relevant literature and research studies, particularly those published since the introduction of the current Primary School Mathematics Curriculum in 1999. In line with the research request, we focused on features of good pedagogy as they apply to all children, including exceptional children, children in culturally diverse contexts and children in disadvantaged circumstances. We reviewed research related to curriculum design and presentation, and the specification of goals related to processes and content. We have given attention to research on language, integration, time, working with parents, and teacher preparation and development.

The implications for curriculum development presented here are based on a view of curriculum as being multi-faceted. It comprises documentation in which aims, goals and teaching activities are explicated. However, it also involves what happens in classrooms, i.e., what children learn. There needs to be a good fit between these levels. This means that educators need to work together in interrogating the curriculum and negotiating it at a local level.

Our implications are presented in a context in which there is a growing awareness of the extent of mathematical learning in the pre-school years and its significance for later development. Important contextual factors include developments in preschool provision, the increased involvement of parents in their children's education, the multicultural nature of children's learning environments, the ever-growing presence of technology in all aspects of children's lives, concerns about children's mathematical achievements and attitudes, and an economy in which mathematical knowledge is increasingly valued.

The key implications for the redevelopment of the mathematics curriculum arising from this review of research presented in this report are as follows:

- The curriculum should be coherent in terms of aims and goals relating to both processes and content, and pedagogy.

- The processes of mathematization, that is, communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting, should be foregrounded in curriculum documentation and should be central to the mathematical experiences of all children.
- The redeveloped mathematics curriculum needs to acknowledge and build on the pedagogical emphases in *Aistear.*
- In order to facilitate transition, educators across early education settings need to communicate about children's mathematical experiences and the features of pedagogy that support children's learning.
- The principles and features of good mathematics pedagogy as they pertain to people and relationships, the learning environment, and the learner, should be emphasised.
- The overarching meta-practices – math talk, productive disposition, modeling, cognitively challenging tasks, and formative assessment – and the ways in which they permeate everyday practices (e.g., story/picture-book reading and project work) should be clearly explicated.
- Educators should be supported in the design and development of rich and challenging mathematical tasks that are appropriate to their children's learning needs.
- The curriculum should exemplify how tools, including digital tools, can enhance mathematics learning.
- Children should engage with all five content domains – Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance. The strand of Early Mathematical Activities as presented in the current PSMC should be integrated into the five content areas.
- In the curriculum documentation, critical ideas in each content domain need to be explicated. These critical ideas, derived from learning paths, should serve as reference points for planning and assessment. In presenting these ideas, over-specification should be avoided. Learning outcomes arising from these also need to be articulated.
- Narrative descriptors of mathematical development, that is, descriptions of critical ideas, should be developed in class bands, e.g., two years. These critical ideas indicate shifts in children's mathematical reasoning in each of the content domains.
- The principles of equity and access should underpin the redeveloped mathematics curriculum. The nature of support that enables exceptional children, children in culturally diverse contexts and children in disadvantaged circumstances to experience rich and engaging mathematics should be specified.
- Intervention for children at risk of mathematical difficulties should begin at a much earlier point than is specified in current guidelines.

- Learning outcomes in mathematics should be cross-referenced with other areas of learning and vice-versa, in order to facilitate integration across the curriculum.
- Time allocated for mathematics should reflect the increased emphases on mathematization and its associated processes.
- Ongoing communication and dialogue with parents and the wider community should focus on the importance of mathematics learning in the early years, the goals of the mathematics curriculum and ways in which children can be supported to achieve these goals.
- Structures should be put in place that encourage and enable the development of mathematical knowledge for pre-service and in-service teachers. Educators need to be informed about goals, learning paths and critical ideas. Records of practice, to be used as a basis for inquiry into children's mathematical learning and thinking, need to be developed.
- Educators need to be given opportunities to interrogate and negotiate the redeveloped curriculum with colleagues as it relates to their setting and context. Time needs to be made available to educators to engage in collaborative practices such as lesson study.
- Given the complexities involved, it is imperative that all educators of children aged 3–8 years develop the knowledge, skills, and dispositions required to teach mathematics well.
- Given the central importance of mathematics learning in early childhood and as a foundation for later development, mathematics should be accorded a high priority, at both policy and school levels, similar to that accorded to literacy.

GLOSSARY

Glossary

Argumentation
'a social phenomenon; when the cooperating individuals try to adjust their intentions and interpretations by verbally presenting the rationale of their actions' (Krummheuer, 1995, p. 229).

Classroom
refers to any group setting for 3- to 8-year-old children (e.g., preschool, family child care, primary school) (NAEYC/NCTM, 2010/2012).

Communicating mathematically
there are a number of ways that children can communicate in mathematics, including oral, visual, digital, textual and symbolic.

Connecting
the notion of 'connections' in mathematics relates both to those that exist (i) within and between different content areas in mathematics (e.g., within number or between number and measurement), (ii) between mathematics learning and learning in other areas and (iii) between mathematics and the context within which a child lives, works or plays (Perry & Dockett, 2008).

Critical transitions
are key developmental understandings related to that concept or domain that are essential for children's understanding of a particular concept or domain.

Curriculum development
the two levels of curriculum development are (a) conceptualisation of plans and the development of resources for teachers and (b) what teachers 'do' to implement these plans in their classrooms (Remillard, 1999).

Exceptional children
children with developmental delays or who are especially talented at mathematics.

Generalising
involves a shift in thinking from specific statements to more general assertions. Children begin to generalise from an early age – for example in learning to use the term 'cup' to refer to all cups (Mason, 2008).

Intentional teaching
the skill of adapting teaching to the content, type of learning experience, and individual child with a clear learning target as a goal (NRC, 2009, p. 226).

Justifying
can be thought of in terms of self-explanation, which is described as 'inferences concerning 'how' and 'why' events happen' (Siegler and Lin, 2010, p. 85).

Knowledge Quartet
a framework devised to aid the development of mathematical knowledge for teaching. Arising from studying primary mathematics teaching, the Knowledge Quartet identifies four dimensions along which teachers' mathematical knowledge impacts on teaching: foundation, transformation, connection and contingency (Rowland et al, 2005).

Learning outcomes
expected outcomes related to children's mathematical learning.

Learning paths
sequences that apply in a general sense to children's development in the different domains of mathematics.

Math talk
children talking about mathematical thinking and engaging in reasoning, argumentation, justification etc.

Mathematical goals
relate to processes and to content. In the literature, the idea of goals is often conflated with the notion of 'big ideas'.

Mathematical Knowledge for Teaching
the construct of Mathematical Knowledge for Teaching (MKT) is claimed to conceptualise the specialised knowledge teachers need in order to complete the tasks of teaching mathematics. MKT is thought to consist of Subject Matter Knowledge and Pedagogical Content Knowledge (Ball & Bass, 2003).

Mathematical modeling
from the RME perspective, modeling is seen as an a organising activity from which a model emerges (Gravemeijer & Stephan, 2002, p. 148).

Mathematical task
an activity, the purpose of which is to focus students' attention on a particular mathematical idea (Stein et al., 1996, p. 460).

Meta-practices
overarching practices which characterise good mathematics pedagogy (promotion of math talk, development of a positive disposition, emphasis on mathematical modeling, use of cognitively challenging tasks, and formative assessment).

Modeling problems
realistically complex situations where the problem-solver engages in mathematical thinking beyond the usual school experience and where the products to be generated often include complex artefacts or conceptual tools that are needed for some purpose, or to accomplish some goal (English & Sriraman, 2010, p. 173).

Narrative descriptors
descriptors of critical ideas in each content domain. These indicate shifts in mathematical thinking at key transitions.

Number sense
characterised as a holistic concept of quantitative intuition, or a feel for numbers and their interrelationships.

Pedagogy
the deliberate process of cultivating development (Bowman, Donovan & Burns, 2001, p. 182); the practice or the art, the science or the craft of teaching (Siraj-Blatchford et et al., 2002, p. 27).

Project
an in-depth study of a particular topic undertaken by small groups of children (Katz & Chard, 2000).

Reasoning
generally associated with logic and the drawing of valid conclusions (e.g., Artzt & Yaloz-Femia, 1999; Steen, 1999).

Records of practice
multimedia case studies or written episodes of classroom vignettes that help educators to situate their own mathematics learning in an authentic classroom experience.

Representing
among the forms of representation that children use to organise and convey their thinking are concrete manipulatives, mental models, symbolic notation, tables, graphs, number lines, stories, and drawings (Langrall et al., 2008). These are sometimes referred to in the literature as 'tools' (see below).

Tools
refer both to physical artefacts and symbolic resources. Physical artefacts include manipulative materials, pens, books and computers, while symbolic resources include language, drawings and diagrams (Armstrong et al., 2005).

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National Council for Curriculum and Assessment

ISSN 1649-3362

National Council for Curriculum and Assessment
35 Fitzwilliam Square, Dublin 2.

T: +353 1 661 7177
F: +353 1 661 7180
E: info@ncca.ie

www.ncca.ie

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Mathematics in Early Childhood and Primary Education (3-8 years)

Executive Summaries

NCCA
An Chomhairle Náisiúnta Curaclaim agus Measúnachta
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F: +353 1 661 7180
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Preface

The mathematical literacy of our children and young people is key to their participation in learning and education, and to their future life chances and employment opportunities. Increasingly, a high standard of mathematical skills generally is an important element in Ireland's economic development. *Literacy and numeracy for learning and life,* Ireland's national strategy to improve literacy and numeracy among children and young people, acknowledges the importance of mathematics and presents a shared goal for numeracy for parents and communities; practitioners and teachers; and leadership in schools.

The Project Maths initiative, which began in post-primary schools in 2008, emphasises the development of conceptual understanding, reasoning and problem solving skills. Since the development of mathematical concepts begins very early in a child's education it makes sense that we turn our attention now to *what* a child learns in mathematics and *how*, beginning with the early years of primary school. *Aistear: the Early Childhood Curriculum Framework* (2009) also highlights the potential and promise of a more child-centred approach to the development of children's early mathematical literacy.

This booklet contains Executive Summaries of two research reports which the NCCA commissioned to support the development of the Primary Mathematics Curriculum:

- *Mathematics in Early Childhood and Primary Education (3-8 years), Definitions, Theories, Development and Progression*
- *Mathematics in Early Childhood and Primary Education (3-8 years), Teaching and Learning.*

The contents of the full reports, which are available at *ncca.ie/primarymaths*, serve to enliven and enlighten our understanding and discussion of children's mathematical learning and development in the early childhood and primary years, and the kinds of curriculum and assessment supports needed. In order to broaden access to key messages from the reports, the authors have also prepared a series of short podcasts (available at *ncca.ie/primarymaths*) in which they discuss important ideas in the reports for parents, practitioners and teachers. The authors are to be commended on these excellent reports which deepen and enrich the context for work on the Primary Mathematics Curriculum.

The NCCA is committed to quality in developing curriculum and assessment which is both evidence-based and informed by practice. These research reports mark the beginning of Council's work to develop the new mathematics curriculum for primary schools. We look forward to a wide-ranging engagement with all concerned in this important task.

Brigid McManus
Chairperson, NCCA

Table of Contents

The executive summaries of reports No. 17 and No. 18 are available online at *ncca.ie/primarymaths*. The online versions include some hyperlinks which appear as text on dotted lines in this print copy.

RESEARCH REPORT NO. 17

Definitions, Theories, Development and Progression

The review of research on mathematics learning of children aged 3–8 years is presented in two reports. These are part of the NCCA's Research Report Series (ISSN 1649–3362). The first report (Research Report No. 17) focuses on theoretical aspects underpinning the development of mathematics education for young children. The second report (Research Report No. 18) is concerned with related pedagogical implications. The key messages from Report No. 17 are presented in this Executive Summary.

A View of Mathematics

Both reports are underpinned by a view of mathematics espoused by Hersh (1997). That is, mathematics as 'a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context' (p. xi). Mathematics is viewed not only as useful and as a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004). All children are viewed as having an ability to solve mathematical problems, make sense of the world using mathematics, and communicate their mathematical thinking.

Context

The context in which this report is presented is one in which there is a growing awareness of the importance of mathematics in the lives of individuals, in the economy and in society more generally. In parallel with this there is a growing realisation of the importance of the early childhood years as a time when children engage with many aspects of mathematics, both at home and in educational settings (Ginsburg & Seo, 1999; Perry & Dockett, 2008). Provision for early childhood education in Ireland has also increased. A recent development is free preschool education for all children in the year prior to school entry. In addition, a new curriculum framework, *Aistear* (National Council for Curriculum and Assessment [NCCA], 2009a; 2009b), is available to support adults in developing children's learning from birth to six years. At the same time, however, there are concerns about the levels of mathematical reasoning and problem-solving amongst school-going children, as evidenced in recent national and international assessments and evaluations at primary and post-primary levels (e.g., Eivers et al., 2010; Perkins, Cosgrove, Moran & Shiel, 2012; Jeffes et al., 2012). While the 1999 Primary School Mathematics Curriculum (PSMC) has been well received by teachers (NCCA, 2005), the Inspectorate of the then Department of Education and Science identified some difficulties with specific aspects of implementation (DES, 2005). The current report envisions a revised PSMC that is responsive to these concerns, that recognises the importance of building on children's early engagement with mathematics, and which takes account of the changing demographic profile of many educational settings, and the increased diversity among young children.

Definitions of Mathematics Education

Current views of mathematics education are inextricably linked with ideas about equity and access and with the vision that mathematics is for all (Bishop & Forganz, 2007), i.e. all children should have opportunities to engage with and benefit from mathematics education and no child should be excluded.

Mathematics education is seen as comprising a number of mathematical practices that are negotiated by the learner and teacher within broader social, political and cultural contexts (Valero, 2009). An interpretation of mathematics that includes numeracy but is broader should underpin efforts towards curricular reform in

Ireland. This report identifies mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition) (NRC, 2001) as a key aim of mathematics education. It is promoted through engagement with processes such as connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. All of these are encompassed in the overarching concept of mathematization. This involves children interpreting and expressing their everyday experiences in mathematical form and analysing real world problems in a mathematical way through engaging in these key processes (Ginsburg, 2009a; Treffers & Beishuizen, 1999). Thus mathematization is identified as a key focus of mathematics education and as such it is given considerable attention in this report. Mathematics education should address the range of mathematical ideas that all children need to engage with. It should not be limited to number.

Theoretical Perspectives

Cognitive and sociocultural perspectives provide different lenses with which to view mathematics learning and the pedagogy that can support it (Cobb, 2007). Cognitive perspectives are helpful in focusing on individual learners while sociocultural perspectives are appropriate when focusing on, for example, pedagogy (Cobb & Yackel, 1996). Sociocultural, cognitive perspectives and constructionism all offer insights which can enrich our understanding of issues related to the revision of the curriculum. They do so by providing key pointers to each of the elements of learning, teaching, curriculum and assessment. Used together they can help in envisaging a new iteration of the PSMC.

In this report, learning mathematics is presented as an active process which involves meaning making, the development of understanding, the ability to participate in increasingly skilled ways in mathematically-related activities and the development of a mathematical identity (Von Glasersfeld, 1984; Rogoff, 1998; Lave & Wenger, 1991). Learning also involves the effective use of key tools such as language, symbols, materials and images. It is seen to be supported by participation in the community of learners engaged in mathematization, in small-group and whole class conversations. The proactive role of the teacher must be seen to involve the creation of a zone of proximal development, the provision of scaffolding for learning and the co-construction of meaning with the child based on awareness and understanding of the child's perspective (e.g., Bruner, 1996). It also involves a dialogical pedagogy of argumentation and discussion designed to support effective conceptual learning and the ability for teachers to act contingently (e.g., Corcoran, 2012).

Language and Communication

Cognitive/constructivist and sociocultural perspectives on learning emphasise the key role of language in supporting young children's mathematical development. Emerging learning theories point to the importance of mathematical discourse as a tool to learn mathematics (e.g., Sfard, 2007). In addition to introducing young children to mathematical vocabulary, it is important to engage them in 'math talk' – conversations about their mathematical thinking and reasoning (Hufferd-Ackles, Fuson & Sherin, 2004). Such talk should occur across a broad range of contexts, including unplanned and planned mathematics activities and activities such as storytelling or shared reading, where mathematics may be secondary. Children at risk of mathematical difficulties, including those living in disadvantaged circumstances, may need additional, intensive support to develop language and the ability to participate in mathematical discourse (Neuman, Newman & Dwyer, 2011).

Research indicates an association between the quality and frequency of mathematical language used by carers, parents and teachers as they interact with young children, and children's development in important aspects of mathematics (Klibanoff et al., 2006; Gentner, 2003; Levine et al., 2012). This highlights the importance of adults modelling mathematical language and encouraging young children to use such language. Conversations amongst children about mathematical ideas are also important for mathematical development (e.g., NRC, 2009).

Defining Goals

The goal statements of a curriculum should be aligned with its underlying theory. Curriculum goals should reflect new emphases on ways to develop children's mathematical understandings and to foster their identities as mathematicians (Perry & Dockett, 2002; 2008). This report proposes that processes and content should be clearly articulated as related goals (e.g., mathematization can be regarded as both a process and as content since as children engage in processes e.g., connecting, they construct new and/or deeper understandings of content). This contrasts with the design of the Primary School Mathematics Curriculum (PSMC), where content and processes are presented separately, and content is emphasised over processes. An approach in which processes are foregrounded, but content areas are also specified, is consistent with a participatory approach to mathematics learning and development.

General goals need to be broken down for planning, teaching and assessment purposes. This can be done through identifying critical ideas i.e., the shifts in mathematical reasoning required for the development of mathematical concepts (e.g., Simon, 2006; Sarama & Clements, 2009). An understanding of this framework enables teachers to provide support for children's progression towards curriculum goals.

The Development of Children's Mathematical Thinking

The idea of stages of development in children's mathematical learning (most often associated with Piaget) has now been replaced with ideas about developmental/learning paths. This is a relatively recent area of research in mathematics education

(Daro et al., 2011) and as such is still under development. Learning paths are also referred to as learning trajectories. They indicate the sequences that apply in a general sense to development in the various domains of mathematics (e.g., Fosnot & Dolk, 2001; Sarama & Clements, 2009; van den Heuvel-Panhuizen, 2008). This report envisages that general learning paths will provide teachers with a basis for assessing and interpreting the mathematical development in their own classroom contexts, and will lead to learning experiences matched to individual children's needs.

There is variation in the explication of learning paths, for example, linear/nonlinear presentation, level of detail specified, mapping of paths to age/grade, and role of teaching. Different presentations reflect different theoretical perspectives. An approach to the specification of learning paths that is consistent with sociocultural perspectives is one which recognises the paths as

i. provisional, as many children develop concepts along different paths and there can never be certainty about the exact learning path that individual children will follow as they develop concepts

ii. not linked to age, since this suggests a normative view of mathematics learning

iii. emerging from engagement in mathematical-rich activity with children reasoning in, and contributing to, the learning/teaching situation (e.g., Fosnot & Dolk, 2001; Stigler & Thompson, 2012; Wager & Carpenter, 2012).

Assessing and Planning for Progression

Of the assessment approaches available, formative assessment offers most promise for generating a rich picture of young children's mathematical learning (e.g., NCCA, 2009b; Carr & Lee, 2012). Strong conceptual frameworks are important for supporting teachers' formative assessments (Carr & Lee, 2012; Ginsburg, 2009a; Sarama & Clements, 2009). These influence what teachers recognise as significant learning, what they take note of and what aspects of children's activity they give feedback on. There is a range of methods (observation, tasks, interviews, conversations, pedagogical documentation) that can be used by educators to assess and document children's mathematics learning and their growing identities as mathematicians. Digital technologies offer particular potential in this regard. These methods are challenging to implement and require teachers to adopt particular, and for some, new, perspectives on mathematics, mathematics learning and assessment. Constructing assessments which enlist children's agency (for example, selecting pieces for inclusion in a portfolio or choosing particular digital images to tell a learning story) has many benefits. One benefit is the potential for the inclusion of children's perspectives on their learning (Perry & Dockett, 2008).

In the main, the current literature affords scant support for the use of standardised tests with children in the age range 3–8 years (e.g., Mueller, 2011). More structured teacher-initiated approaches and the use of assessment within a diagnostic framework may be required on some occasions, for example, when children are at risk of mathematical difficulties. However, research indicates a range of factors problematising the use of standardised measures with young children (e.g., Snow & Van Hemel, 2008).

The complex variety of language backgrounds of a significant minority of young children presents a challenge in the learning, teaching and assessment of mathematics. Children for whom the language of the home is different to that of the school need particular support. That support should focus on developing language, both general and mathematical, to maximise their opportunities for mathematical development and their meaningful participation in assessment (Tabors, 2008; Wood & Coltman, 1998). Educators carrying out assessment procedures such as interviews, observations or tasks in an immersion context have the dual purpose of assessing and evaluating both the mathematical competences and language competences of the child, to gain a full picture. Dual language assessment is particularly desirable in this context (Murphy & Travers, 2012; Rogers, Lin & Rinaldi, 2011).

Addressing Diversity

Mathematics 'for all' implies a pedagogy that is culturally sensitive and takes account of individuals' ways of interpreting and making sense of mathematics (Malloy, 1999; Fiore, 2012). An issue of concern is the limitations of norms-based testing which can disadvantage certain groups. This indicates the need to use a diverse range of assessment procedures to identify those who are experiencing learning difficulties in mathematics.

The groups of individuals that often require particular attention in the teaching and learning of mathematics are 'exceptional' children (those with developmental disabilities or who are especially talented at mathematics) (Kirk, Gallagher, Coleman, & Anastasiow, 2012). These individuals do not require distinctive teaching approaches, but there is a need to address their individual needs. In particular, the use of multi-tiered tasks in which different levels of challenge are incorporated is advocated (Fiore, 2012).

In addition, this report identifies the need to provide parents and educators with particular supports to ensure a mathematically-interactive and rich environment for children aged 3–8 years. It also indicates that the intensity of the support needs to vary according to the needs of particular groups of children (e.g., Ehrlich, Levine, & Goldin-Meadow, 2006).

Key Implications

The following are the key implications that arise from this report for the development of the mathematics curriculum for children aged 3–8 years:

- In the curriculum, a view of all children as having the capacity to engage with deep and challenging mathematical ideas and processes from birth should be presented. From this perspective, and in order to address on-going concerns about mathematics at school level, a curriculum for 3–8 year-old children is critical. This curriculum needs to take account of the different educational settings that children experience during these years.

- The curriculum should be developed on the basis of conversations amongst all educators, including those involved in the NCCA's consultative structures and processes, about the nature of mathematics and what it means for young children to engage in doing mathematics. These conversations should be informed by current research, as synthesised in this report and in Report No. 18, which presents a view of mathematics as a human activity that develops in response to everyday problems.

- The overall aim of the curriculum should be the development of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition). As mathematization plays a central role in developing proficiency, the processes of mathematization should permeate all learning and teaching activities. These include connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. (*Chapter 1*)

- The curriculum should foreground mathematics learning and development as being dependent on children's active participation in social and cultural experiences, while also recognising the role of internal processes. This perspective on learning provides a powerful theoretical framework for mathematics education for young children. Such a framework requires careful explication in the curriculum and its implications for pedagogy should be clearly communicated. (*Chapter 2*)

- In line with the theoretical framework underpinning the curriculum, mathematical discourse (math talk) should be integral to the learning and teaching process. The curriculum should also promote the development of children's mathematical language in learning situations where mathematics development may not be the primary goal. Particular attention should be given to providing intensive language support, including mathematical language, to children at risk of mathematical difficulties. (*Chapter 3*)

- The goal statements of the curriculum should be aligned with its underlying theory. An approach whereby processes are foregrounded but content areas are also specified is consistent with a participatory approach to mathematics learning and development. In the curriculum, general goals need to be broken down for planning, teaching and assessment purposes. Critical ideas indicating the shifts in mathematical reasoning required for the development of key concepts should be identified. (*Chapter 4*)
- Based on the research which indicates that teachers' understanding of developmental progressions (learning paths) can help them with planning, educators should have access to information on general learning paths for the different domains. Any specification of learning paths should be consistent with sociocultural perspectives, which recognise the paths as provisional, non-linear, not age-related and strongly connected to children's engagement in mathematically-rich activity. Account needs to be taken of this in curriculum materials. Particular attention should be given to the provision of examples of practice, which can facilitate children's progression in mathematical thinking. (*Chapter 5*)
- The curriculum should foreground formative assessment as the main approach for assessing young children's mathematical learning, with particular emphasis on children's exercise of agency and their growing identities as mathematicians. Digital technologies offer particular potential in relation to these aspects of development. The appropriate use of screening/diagnostic tests should be emphasised as should the limitations of the use of standardised tests with young children. The curriculum should recognise the complex variety of language backgrounds of a significant minority of young children and should seek to maximise their meaningful participation in assessment. (*Chapter 6*)

- A key tenet of the curriculum should be the principle of 'mathematics for all'. Central to this is the vision of a multicultural curriculum which values the many ways in which children make sense of mathematics. While there are some groups or individuals who need particular supports in order to enhance their engagement with mathematics, in general distinct curricula should not be advocated. (*Chapter 7*)
- Curriculum developments of the nature described above are strongly contingent on concomitant developments in pre-service and in-service education for educators at preschool and primary levels.

RESEARCH REPORT NO. 18

Teaching and Learning

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A View of Mathematics

Both volumes are underpinned by a view of mathematics espoused by Hersh (1997): mathematics as 'a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context' (p. xi). Mathematics is viewed not only as useful and as a way of thinking, seeing and organising the world, but also as aesthetic and worthy of pursuit in its own right (Zevenbergen, Dole, & Wright, 2004). All children are viewed as having an ability to solve mathematical problems, make sense of the world using mathematics, and communicate their mathematical thinking. This shift in perspective demands a change in pedagogy – in particular it puts the teaching-learning relationship at the heart of mathematics.

Context

In Report No. 17 we argue that the overall aim of the curriculum should be the development of mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition) (National Research Council [NRC], 2001). As mathematization plays a central role in developing proficiency, the processes of mathematization should permeate all learning and teaching activities. These include connecting, communicating, reasoning, argumentation, justifying, representing, problem-solving and generalising. Foregrounding mathematical proficiency as the aim of mathematics education has the potential to change the kind of mathematics and mathematical learning that young children experience. As it demands significant changes in pedagogy, curriculum and curricular supports (Anthony & Walshaw, 2007), it also poses challenges that are wide-ranging and systemic.

The development of mathematical proficiency begins in the preschool years, and individuals become increasingly mathematically proficient over their years in educational settings. This implies that educators in the range of early childhood settings need to develop effective pedagogical practices that engage learners in high-quality mathematics experiences. There is a concomitant need to address issues related to curriculum content and presentation. In particular, the questions of how to develop a coherent curriculum and how to formulate progressions in key aspects of mathematics are important. The view of curriculum presented in this report is both wide and dynamic. It is recognised that the mathematics education of young children extends beyond the walls of the classroom: family and the wider community can make a significant contribution to children's mathematical achievement (e.g., Sheldon & Epstein, 2005).

Pedagogy

It is impossible to think about good mathematics pedagogy for children aged 3–8 years without acknowledging that much early mathematical learning occurs in the context of children's play (e.g., Seo & Ginsburg, 2004). Educators need to understand how mathematics learning is promoted by young children's engagement in play, and how best they can support that learning. For instance, adults can help children to maximise their learning by helping them to represent and reflect on their experiences (e.g., Perry & Dockett, 2007a). Learning through play is seen as fundamental to good mathematics pedagogy in early childhood. It assumes varying degrees of emphasis depending on the age of the child. Recent research points to a number of other important principles which underpin good mathematics pedagogy for children aged 3–8 years (e.g., Anthony & Walshaw, 2009a; NRC, 2005). These principles focus on people and relationships, the learning environment and learners. Features of good mathematics pedagogy can be identified with reference to these principles. Both the principles and the features of pedagogy are consistent with the aim of helping children to develop mathematical proficiency. They pertain to all early educational settings, and are important in promoting continuity in pedagogical approaches across settings.

Practices

Good mathematics pedagogy incorporates a number of meta-practices (i.e., overarching practices) including the promotion of math talk, the development of a productive disposition, an emphasis on mathematical modeling, the use of cognitively challenging tasks, and formative assessment. The literature offers a range of perspectives, and advice, as to the issues for educators in integrating these elements into their practices. In doing so, the vision of 'mathematics for all' is supported.

Good mathematics pedagogy can be enacted when educators engage children in a variety of mathematically-related activities across different areas of learning. The activities should arise from children's interests, questions, concerns and everyday experiences. A deep understanding of the features of good pedagogy should inform the ways in which educators engage children in mathematically-related activities such as play, story/picture-book reading, project work, the arts and physical education. The potential of these activities for developing mathematical proficiency can best be realised when educators focus on children's mathematical sense-making. In addition, educators need to maximise the opportunities afforded by a range of tools, including digital tools, to mediate learning.

Curriculum Development

Goals, coherent with the aim of mathematical proficiency, should be identified. These goals relate both to process and content. The processes of mathematization, that is, communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting, should be foregrounded. In line with the principle of 'mathematics for all', each of the five content domains – Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance – should be given appropriate attention.

Goals need to be broken down for planning, teaching and assessment purposes. Learning paths can be helpful for this purpose. As is outlined in Report No. 17, differences in the ways learning paths are presented in the literature rest largely on their theoretical underpinnings. For example, developmental progressions described by Sarama and Clements (2009) are finely grained and age-related, whereas the TAL[1] trajectories developed in the context of Realistic Mathematics Education (van den Heuvel-Panhuizen, 2008) are characterised by fluidity and the role of context. In line with a sociocultural approach to the learning of mathematics, we advocate that learning paths be used in a flexible way to posit shifts in mathematical reasoning, i.e. critical ideas in each of the domains. Narrative descriptors of critical ideas can be used to inform planning and assessment. Learning outcomes, relating to content domains and processes, can then be derived from a consideration of the goals, learning paths and narrative descriptors. The figure below shows an emerging curriculum model highlighting how the relationships between the different elements may be conceptualised.

1 In Dutch, learning-teaching trajectories are referred to as TALs (i.e., Tussendoelen Annex Leerlijinen).

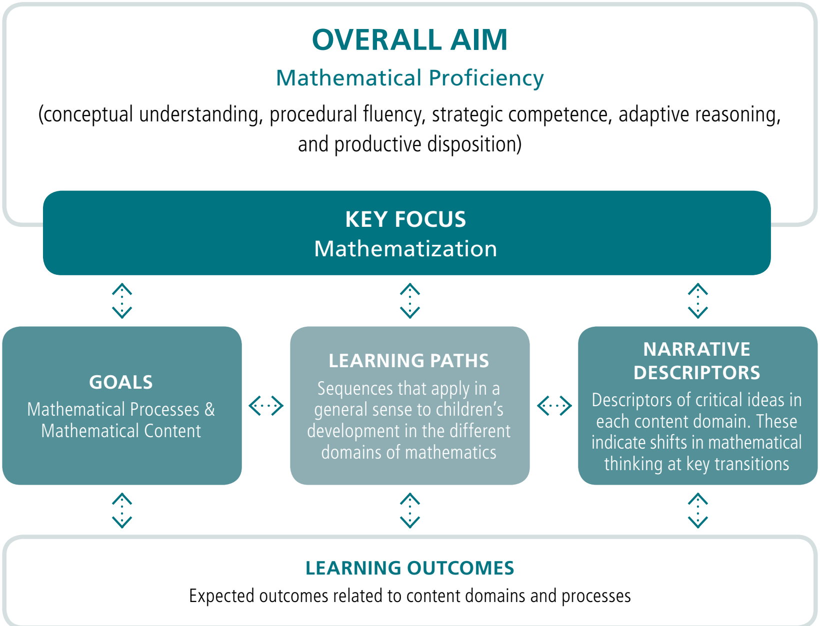

Figure ES.1: Emerging Curriculum Model

Curricular Issues

While the specification of processes and content in the mathematics curriculum is critically important, attention should also be given to issues that relate to curriculum access and curriculum implementation. This is based on the premise that the curriculum must serve all children, including exceptional children (those with developmental delays and those with exceptional talent) and children in culturally diverse contexts. Other key issues include the timing of early intervention, the allocation of time to mathematics in early learning settings, and how best to achieve the integration of mathematics across the curriculum.

Consistent with Lewis and Norwich's (2005) concept of continua of common teaching approaches that can be subject to varying degrees of intensity depending on children's needs, modifications to the mathematics curriculum for children with special education needs are proposed. Mathematically-talented children should be supported in deepening their understanding of and engagement with the existing curriculum rather than being provided with an alternative one. In the case of English-language learners, and children attending Irish-medium schools, the key role of mathematical discourse and associated strategies in enabling access to the language in which the curriculum is taught are emphasised (e.g., Chapin, O'Connor, & Anderson, 2009). Attention to language is also highlighted as a critical issue in raising the mathematics achievement of children in DEIS schools. More generally, it is noted that there is now strong research indicating that additional support should be provided at an earlier stage than is indicated in current policy documents (e.g. Dowker, 2004; 2009). There is a need to allocate sustained time to mathematics to ensure that all children engage in mathematization. Dedicated and integrated time provision is recommended. The value of integrating mathematics across areas of learning is recognised, though it is acknowledged that relatively little research is available on how best to achieve this.

Partnership with Parents

In line with the emphasis on parental involvement in the *National Strategy to Improve Literacy and Numeracy among Children and Young People (2011–2020)* (Department of Education and Skills [DES], 2011a), the key role of parents in supporting children to engage in mathematics is emphasised. There is a range of activities in which parents can engage with schools so that both parents and educators better understand children's mathematics learning. However, it is acknowledged that research on parental involvement in mathematics lags behind similar research relating to parental involvement in reading literacy.

In the literature on parental involvement, the need to establish a continuous, two-way flow of information about children's mathematics learning between educators and parents is a key theme. There is potential for technology to support this. Strategies designed to support parents to better understand their child's mathematical learning include observation of and discussion on children's engagement in mathematical activities in education settings. Mechanisms are required to inform parents about the importance of mathematics learning in the early years, and what constitutes mathematical activity and learning for young children. The significant role that parents play in the mathematical development of their children should be foregrounded.

Teacher Preparation and Development

Curriculum redevelopment is strongly contingent on parallel developments in pre-service and in-service education for educators across the range of settings. In particular, professional development programmes need to focus on the features of good mathematics pedagogy and the important meta-practices that arise from these.

In order for teachers to foster mathematical proficiency in children, they themselves need to be mathematically proficient. Therefore, teacher preparation courses need to provide opportunities for pre-service teachers to engage in rich mathematical tasks. Educators need to develop mathematical knowledge for teaching through a collaborative focus on teaching and learning of mathematics. They need opportunities to notice children's engagement in mathematics and responses to mathematical ideas. Case studies of practice are valuable tools in this regard. These can be used by pre-service (and in-service) teachers to question and critique the practice of others in order to develop 'local knowledge of practice' (Cochran-Smith, 2012, p. 46).

Among the recommendations for the continuing professional development of teachers (CPD) is investment in stronger systems of clinical supervision across the preparation-induction boundary (Grossman, 2010). The notion of clinical supervision could mean an emphasis on developing good mathematics teaching practices through collaborative review and reflection on existing practice. This is important because inquiry as a stance has been advocated as a successful key to teacher change (Jaworski, 2006). In this regard, lesson study is a practice that is currently foregrounded in the literature as a significant development in school-based professional development (e.g., Corcoran & Pepperell, 2011; Fernández, 2005). In lesson study, publicly available records of practice or 'actionable artifacts' are important by-products (Lewis, Perry, & Murata, 2006, p. 6). The practice offers opportunities at school and classroom level for enactment of critical inquiry into mathematics lessons.

Key Implications

The key implications for the redevelopment of the mathematics curriculum arising from the review of research presented in this report are as follows:

- The curriculum should be coherent in terms of aims, goals relating to both processes and content, and pedagogy. (*Chapter 1*, *Chapter 3*)
- The processes of mathematization, that is, communicating, reasoning, argumentation, justifying, generalising, representing, problem-solving, and connecting, should be foregrounded in curriculum documentation and should be central to the mathematical experiences of all children. (*Chapter 2*, *Chapter 3*)
- The redeveloped mathematics curriculum needs to acknowledge and build on the pedagogical emphases in *Aistear*. (*Chapter 2*)
- In order to facilitate transitions, educators across early education settings need to communicate about children's mathematical experiences and the

features of pedagogy that support children's learning. (*Chapter 1*)

- The principles and features of good mathematics pedagogy as they pertain to people and relationships, the learning environment, and the learner, should be emphasised. (*Chapter 1*)
- The overarching meta-practices and the ways in which they permeate learning activities should be clearly explicated. (*Chapter 2*)
- Educators should be supported in the design and development of rich and challenging mathematical tasks that are appropriate to their children's learning needs. (*Chapter 2*, *Chapter 5*)
- The curriculum should exemplify how tools, including digital tools, can enhance mathematics learning. (*Chapter 2*)
- Children should engage with all five content domains – Number, Measurement, Geometry and Spatial Thinking, Algebraic Thinking, and Data and Chance. The strand of Early Mathematical Activities as presented in the current PSMC should be integrated into the five content areas. (*Chapter 3*)
- In curriculum documentation, critical ideas in each content domain need to be explicated and expressed as narrative descriptors. These critical ideas, derived from learning paths, should serve as reference points for planning and assessment. In presenting these ideas, over-specification should be avoided. Learning outcomes arising from these also need to be articulated. (*Chapter 3*)
- Narrative descriptors of mathematical development, that is, descriptions of critical ideas, should be developed in class bands, e.g., two years. These critical ideas indicate shifts in children's mathematical reasoning in each of the content domains. (*Chapter 3*)
- The principles of equity and access should underpin the redeveloped mathematics curriculum. The nature of support that enables exceptional

children (those with developmental delays and those with exceptional talent), children in culturally diverse contexts and children in disadvantaged circumstances to experience rich and engaging mathematics should be specified. (*Chapter 4*)

- Additional support/intervention for children at risk of mathematical difficulties should begin at a much earlier point than is specified in the current guidelines. (*Chapter 4*)
- Learning outcomes in mathematics should be cross-referenced with other areas of learning and vice-versa, in order to facilitate integration across the curriculum. (*Chapter 2*, *Chapter 4*)
- Additional time allocated for mathematics should reflect the increased emphases on mathematization and its associated processes. Some of this additional time might result from integration of mathematics across areas of learning. While integration has the potential to develop deep mathematical understanding, the challenges that it poses to teachers must be recognised. (*Chapter 3*, *Chapter 4*)
- Ongoing communication and dialogue with parents and the wider community should focus on the importance of mathematics learning in the early years, the goals of the mathematics curriculum and ways in which children can be supported to achieve these goals. (*Chapter 5*)
- Structures should be put in place that encourage and enable the development of mathematical knowledge for pre-service and in-service teachers. Educators need to be informed about goals, learning paths and critical ideas. Records of practice, to be used as a basis for inquiry into children's mathematical learning and thinking, need to be developed. (*Chapter 6*)
- Educators need to be given opportunities to interrogate and negotiate the redeveloped curriculum with colleagues as it relates to their setting and

context. Time needs to be made available to educators to engage in collaborative practices such as lesson study. (*Chapter 6*)

- Given the complexities involved, it is imperative that all educators of children aged 3–8 years develop the knowledge, skills, and dispositions required to teach mathematics well. (*Chapter 6*)
- Given the central importance of mathematics learning in early childhood and as a foundation for later development, mathematics should be accorded a high priority, at both policy and school levels, similar to that accorded to literacy. (*Chapter 4*, *Chapter 5*)

NCCA

An Chomhairle Náisiúnta Curaclaim agus Measúnachta
National Council for Curriculum and Assessment

ISSN 1649-3362

National Council for Curriculum and Assessment
35 Fitzwilliam Square, Dublin 2.

T: +353 1 661 7177
F: +353 1 661 7180
E: info@ncca.ie

www.ncca.ie

Design by www.principle.ie

[home]

Audit of mathematics curriculum policy across 12 jurisdictions

Commissioned Report

Authored by Damien Burke, Marino Institute of Education

NCCA

Contents

Executive Summary

A Summary of Key Findings

- It is now timely to review and revise our mathematics curriculum.
- Our curriculum structure and banding arrangements are typical of international curricula.
- Ireland's current five mathematical strands are typical of international curricula.
- By international comparisons, Ireland has a limited range of contemporary curriculum supports.
- Ireland lags behind other countries in their articulation of attainment expectations and provision of illustrative work samples/exemplars.

Mathematics Curricula around the world, have recently undergone substantial revisions, and improvements. These revisions take account of new emphases in mathematics education and assessment – it is now timely for NCCA to take account of the changed educational landscape and begin a review of our current primary mathematics curriculum.

A positive starting point for our 1999 Mathematics Curriculum is the homogeneity between our curriculum structure and banding arrangements, when compared to those of other high-achieving mathematics curricula. Ireland's year by year system of progression in mathematics learning, has obvious benefits when it comes to the provision of more age appropriate content, and facilitation of year on year longitudinal assessment.

Similar to Ireland, a 'Strand-based' structure is evident in the vast majority of curricula. When considering the content of our five curricular strands, and contrasting with those of our neighbours, there is considerable uniformity in terms of what we teach: *Number*, *Measures*, *Geometry* and *Data* can be considered ubiquitous across all jurisdictions. Whilst Ireland specifies *Algebra* as a distinct strand, many other countries choose to integrate it with *Number*. Some countries have chosen to specify particular non-content strands. These strands are typically aimed at prioritising much vaunted higher-order thinking skills amongst pupils, and can certainly be seen as significant contributors to increased levels of attainment in these countries. Such higher order skills are specifically addressed in our own Mathematics Curriculum through the promotion of the six core skills: *Reasoning; Integrating & Connecting; Communicating and Expressing; Implementing; Understanding and Recalling; and Problem-Solving.*

Whilst NCCA and the PDST (Professional Development Service for Teachers) have delivered a huge amount of curriculum supports for teachers to help implement all aspects of the curriculum effectively, Ireland still has some ground to make up. Encouragingly, the sheer scale and variety of international supports available gives us much good practice that we can learn from, some examples include: focused resource hubs and dedicated on-line libraries, teacher video demonstrations, detailed assessment guidance and exemplars, research and relevant industry connections linking in to mathematics teaching at all levels, and finally, state-sponsored teacher networking/learning communities.

One of the main strengths of our existing Mathematics curriculum is the succinct articulation of content objectives at each of the eight class levels, for each of the five curricular strands. As a further aid to enhanced assessment techniques, the articulation of specific observable learning outcomes, or *Expectations*, is now considered a key component of effective curriculum design. In some cases, these *Expectations* are combined with useful assessment rubrics and adjoining curricular manuals to allow for detailed judgements about the varying abilities of pupils.

This audit reveals the comparative strength of our current mathematics curriculum and the relative homogeneity with other high performing education systems, while simultaneously highlighting possible directions for future amendments. We should not hesitate in combining the best of both aspects - this is the balancing act for future curriculum designers.

Introduction

This paper is a reflective commentary based upon the author's desk-top audit of twelve international primary school mathematics curricula, and a further amalgamated curriculum framework from the United States referred to as the *Common Core State Standards for Mathematics*. It attempts to identify examples of contemporary and innovative curriculum design, including structural and content components, associated teacher supports, and finally, instruments for standards, assessment and planning in classrooms. Points of discussion attempt to compare international trends with current curricular elements in the Irish context, and point towards possible future directions for consideration by the National Council for Curriculum and Assessment.

A broad range of countries were chosen by the NCCA for inclusion in the audit, each with a particular attraction: neighbouring British systems due to their obvious cultural similarities, Scandinavian and other European Nations owing to their traditionally strong showing in international mathematics assessment programmes, North American curricula incorporating their strong emphasis on research-driven approaches, and finally Pacific and Asian countries who appear to continue to maximise attainment whilst exploiting more traditional curriculum content and teaching emphases. Primary school curricula from each of the chosen countries were examined individually under three distinct headings: *Curriculum Structure, Curriculum Content and Supports,* and *Assessment and Standards*. This was followed by an analysis phase which sought to identify common trends, and also to pinpoint unique and exceptional innovation. This analysis forms the bedrock of the paper.

The need for curriculum content to keep pace with the rapid advances in mathematics education is acknowledged by the glut of recently revised or redrawn curriculum statements – all such documents date from the mid or late part of the last decade **(see Table 1)**. Curricula revisions in neighbouring British Isle education systems, more so than others, demonstrate that it is now timely for Ireland to review its primary mathematics curriculum – Northern Ireland (2007), Wales (2008), Scotland (2010) and presently, England (2014). This imperative is further strengthened by the innovative inclusion of mathematically significant skills within the themes of *Exploring and Thinking*, and *Communicating* in *Aistear: the Early Childhood Framework* (2009). This new approach to addressing curriculum content through interconnected themes, provides food for thought for future primary curriculum revisions.

Table 1: Implementation Dates for Selected Mathematics Curricula

Year of Implementation	Jurisdiction
2002	Hong Kong S.A.R.
2004	Finland
2005	Ontario
2006	The Netherlands
2007	Northern Ireland, New Zealand
2008	Wales
2010	Scotland, Common Core Standards
2011	Queensland, Massachusetts
2013	Singapore
2014	England

Curriculum structure

The imperative for curriculum authors to organise content in a logical, coherent and accessible manner is a key challenge. The structure of curricula is typically understood to be the headings of content categories which set out the desired knowledge, understandings, capacities and dispositions of the particular discipline. There is a surprising homogeneity across most curricula studied in the structure (and organisers) exploited **(see Table 2)**. With the exception of the Netherlands, which opts for a number of general *Core Objectives*, most curricula deviate little from the current over-arching "strand-based" Irish structure; both Australia and Scotland have a minimalist 3-strand framework, whilst at the opposing end of the range, the new English document specifies six "*Programme of Study*" elements. Many systems elaborate upon these broad domains with an extensive array of strand sub-divisors: Northern Ireland and Ontario clearly demonstrate that such divisors can often over-complicate an initially simple structure; the writer found such curricula difficult to navigate and suggests that this would pose challenges for an integrated approach to the planning and teaching of mathematics. In terms of presentation, the audit reveals that most countries (including the most-recent revisions) still prefer to specify content (and content objectives) on a year-(grade)-by-year basis (as is the case in Ireland). However, there are some notable exceptions: the obvious multi-year *Key-Stage* structure of British systems, and, New Zealand's unique flexibility, which attempts to pace its pupils through the content at a rate that best meets their needs and capacities. This flexibility is evidenced by a continuum ranging from targeted special education provision in mainstream settings, through to culturally-proofed mathematics curriculum content, and cumulating in the availability of accelerated learning programmes[1] for high achievers in numeracy. Of particular interest to an Irish audience is the fact that the newly revised Mathematics Curriculum in England has lessened the influence of defined Key Stages, instead opting for a more fluid path of progression for pupils.

Table 2: Curriculum Structure & Banding (Excludes Infant/Foundation Band as part of primary education unless specified)

Jurisdiction	Number of Bands	Total Duration	Composition	Primary School Starting Age
England	2	6 years	2+4	5
Finland	3	6 years	2+3+1	7
Hong Kong S.A.R.	6	6 years	1 Year per Band	6

[1] http://nzcurriculum.tki.org.nz/System-of-support/School-initiated-supports/PfS

Jurisdiction	Number of Bands	Total Duration	Composition	Primary School Starting Age
Massachusetts	8	8 years	1 Year per Band	6
New Zealand	4	8 years	2+2+2+2 (May Vary)	5
Northern Ireland	3*	7 years	2+2+3	4/5
The Netherlands	2*	8 years	2+6	4
Ontario	8	8 years	1 Year per Band	6
Queensland/*** **	2*	8 years	1+7	5
Scotland	3*	8 years	2+3+3	5
Singapore	6	6 years	1 Year per Band	6/7
Wales	2*	7 years	3+4	5

* (incl. specified Foundation Stage)

** (In Queensland, the curriculum structure is particular to Mathematics)

*** (In Queensland, unlike the remainder of Australia, 7th Grade is considered a primary school grade)

1. Content strands

Number is omni-present across all systems, although seven jurisdictions elect to combine it with a related domain, typically *Number & Algebra*, as evidenced by Singapore, New Zealand, and Australia. This sets an interesting contrast with the very distinct and separate strands of Number and Algebra, which is a feature of the Irish system. This does raise the issue of whether or not the Irish approach helps or hinders integration within mathematics content itself – (an area examined further on page 14). Finland has combined *Number* with the execution of calculation skills, thus creating a clear inter-dependence between the two: *Number & Calculations*. The 2010 Scottish framework infuses an element of real-world application; *Number, Money & Measures*. Its Curriculum Support Section builds upon this integrated approach across the various exemplars it provides, thus giving strong emphasis to the fact that numeric competency is important because of its real-world applications. *Geometry, Measures and Data*, in one guise or another, are present across all twelve curricula (see Table 3). This is even true in the aforementioned Dutch Core Objectives which, upon closer examination, do have considerable similarities in content when compared to the more traditional "Strand" or "Unit of Work" structure, despite their unusual presentation.

Table 3: Key Content Areas

Content Area	Number of Curricula that include this content area (out of 13)
Number	All
Measures	All
Geometry	All
Algebra (as a stand-alone)	9
Data Handling/Statistics	12
Processes in Mathematics	5
Other additional areas such as Early Mathematical Activities	2

2. A differentiated structure

A notable trend in the more recent curriculum statements (2005 onwards, including the newly published Mathematics Curriculum in England) is the movement towards a differentiated curriculum structure as children progress from pre-school through to upper-primary level. Wales has taken the recent step of re-designing its early childhood phase (*Foundation Level*) into a stand-alone (mathematics) syllabus that seeks to address the learning needs of children at that specific stage of their lives and development, not solely with the objective of building skills that will be applied sometime in the future. It should be noted that Ireland's early childhood curriculum framework, *Aistear* compares favourably in this regard – it's thematic approach and integration of play as a key teaching and learning approach, allows for the early development of mathematically significant competencies such as understanding the meaning and use of numbers, and building a sense of time and other measures. Less ambitiously, but in a similar vein, The United States' *Common Core Standards*, Ontario's Curricular Framework and the newly rolled-out English Mathematics Syllabus all tailor the content domains to particular age groups – interestingly, whilst New Zealand exploits a 3-strand structure for all elementary grades, it is the only one of the twelve nations to specify clear recommendations on what percentage of instructional time should be devoted to some strands, principally *Number*. This recommended time for number decreases as children progress beyond 4th grade, and again at 7th grade and upwards. Hong Kong also makes similar suggestions to teachers but such recommendations do not seem to be key features of more recent teacher support documentation from this country. It would appear that the vast majority of countries, despite their universal mandating of curriculum content, do provide local autonomy to allow schools adjust internal subject time allocations as they see fit to meet pupils' mathematical needs. With a recently increased mathematics (but not internal) time allocation, Ireland plots a middle ground that is in keeping with a minority such as Ontario, Singapore and Finland.

3. Process strands

A final feature of innovative curriculum structure noteworthy of mention is the inclusion of non-content specific focused strands: Singapore's *Mathematical Processes* strand, Finland's *Thinking Skills* domain, The Netherlands' *Mathematical Insight (and Operations)* objective, and although a feature of the now obsolete 1999 Mathematics Curriculum, England's *Using and Applying Mathematics* content areas are all appealing as they allow the teacher to present mathematics as a method of enquiry, an instrument for application, not just a series of procedural competencies underpinned by vast reams of mathematical theory. Other typical components employed by various curriculum-design agencies to broaden the curricular appeal of mathematics include the use of *Standards for Mathematical Practice* (e.g. Construct Viable Arguments and Critique the Reasoning of Others - Common Core Standards), *Core Aims* (e.g. Reason Mathematically – Wales) and *Mathematical Processes* (e.g. Reasoning and Proving - Ontario). Use of verbs such as those highlighted contrast with the rather theory-laden and unattractive presentation of Ontario, Northern Ireland and Hong Kong. Ireland's current dual emphasis on *Skills Development* and *Content Objectives* would appear to compare quite favourably in this regard. In addition, the *Aistear* framework, although thematically-based, does in a manner continue this trend by use of broad over-arching *Aims*, supplemented by more specific *Learning Goals* that encompass specific skills and dispositions.

Curriculum content and supports

The sheer scale of curriculum supports across the international spectrum is awesome – online resource hubs (e.g. England and Northern Ireland), dedicated *i-Tunes* and digital television channels (e.g. Hong Kong, Wales and Australia), discussion forums (such as in Scotland), time-efficient curriculum planning tools (e.g. New Zealand and Australia) and professional learning portals (e.g. Ontario and Massachusetts) all underscore our digital age (see Table 4). Ontario[2] is one of the few systems to furnish video footage of real teachers teaching in real classrooms – the short clips give a realistic basis to the content, and provide a more appealing option for educators searching for new lesson ideas. Interestingly, Massachusetts and New Zealand (along with Ontario) are among a small number who provide online and face-to-face supports to teachers who require content-specific upskilling that seeks to build competence and confidence in using mathematics. The audit revealed a noticeable dearth of mathematical language promotion supports. Although virtually all curriculum documents extol the benefits of developing such a competency, it is again only Ontario that provides supports to enhance "Math-Talk" in primary schools. Most national systems provide a recommended glossary of mathematical terms for pupil and/or teacher use, but the presentation is often remote and detached from the objectives of the syllabus, thus giving the impression of being a mere after-thought. In the case of the new Mathematics Curriculum in England, one suspects that the currently small number of teacher resources will be supplemented handsomely as full implementation is reached in 2016.

Table 4: Range of Teaching Supports Available* per Curriculum

Explanatory Legend**: R.H.:** Resource Hubs/Search Tools, **A.V.:** Audio-Visual Demonstrations, **CSAA:** Curriculum Support Agency Aids, **AG:** Assessment Guidance, **R/IL:** Research/Industry Links, **TNO**: Teacher Networking Opportunities

(*An empty cell merely denotes that such resources were not apparent during the audit, but may exist elsewhere or may have been subsequently added)

	RH	AV	CSAA	AG	R/IL	TNO
England	✓	✓	✓	✓	✓	✓
Finland			✓	✓	✓	✓
Hong Kong S.A.R.	✓	✓	✓	✓	✓	
Massachusetts/Common Core Standards	✓	✓	✓	✓		✓

[2] http://www.eworkshop.on.ca/edu/core.cfm

	RH	AV	CSAA	AG	R/IL	TNO
New Zealand	✓		✓	✓	✓	✓
The Netherlands	✓	✓	✓	✓		✓
Ontario	✓	✓	✓	✓	✓	✓
Queensland	✓	✓	✓	✓	✓	✓
Scotland	✓		✓	✓		✓
Singapore	✓	✓	✓		✓	✓
Wales	✓	✓	✓	✓		

4. Mathematics in immersion settings

Supports for mathematics teaching and learning in immersion settings ranged from generalised references to non-native language learners (such as Massachusetts), to the availability of translations of curriculum-support materials (such as Northern Ireland and Scotland to an extent), with culturally-proofed native language curricula (such as New Zealand) at the other end of the spectrum. A closer analysis of the immersion supports in Wales, due to its obvious similarity with Ireland, revealed a rather ad-hoc collection of Welsh-medium supports (including textbooks, websites, official reports, Dept. of Education support documents and resource hubs) that were quite scatter-gunned in addressing the content objectives of the curriculum. It appeared to this writer that the provision of outdated and predominantly textbook-based supports clearly preceded the revised curriculum drafting process in Wales, and therefore congruence between the two is not evident. For Ireland, real-time development of áiseanna as Gaeilge will be vital to ensure their relevance to the emerging content objectives, and the contemporary needs of teachers.

5. Integration

Integration supports in curriculum handbooks rarely exceed aspirational statements on the importance of integrating mathematics with other subjects. The provision of exemplars, which are typically presented as stand-alone and disjointed case-studies of classroom practice, appear in at least ten of the countries, but are exclusively cross-curricular. The Welsh Curriculum emerges as perhaps the most comprehensive supporter of integrated planning and learning with its *Numeracy and Literacy framework* providing a context (and coherent rationale) for the many exemplars and suggestions it makes. As noted in the 'Assessment and standards' portion of the audit, Northern Ireland's solitary cross-curricular skill of *Using*

Mathematics suggests a similar approach is gaining traction in other parts of the United Kingdom.

6. ICT supports

In general, ICT supports are confined to websites that provide a multitude of digital resources and lesson enrichment activities, many with interactive and feedback capacity. Both Hong Kong and the Netherlands are amongst a very small minority that still provide software downloads for teachers. Websites and applications built upon a gaming concept, but related to curriculum content, emerge as being particularly contemporary; Scotland, in particular, is pioneering this field. In fact, the sheer variety of web-links can be off-putting and curriculum agencies that recommend a small selection of carefully chosen websites (with brief descriptions) do seem to strike a better balance. Entry-restricted online portals (such as *Scootle* in Australia, *iShare* in Singapore, *GLOW* in Scotland, along with the publicly accessible Dutch *WIKI-Wiskunde* Reference Library and *Smart Classrooms* in Queensland) provide gateways to vast arrays of classified digital aids that are devised, rated and recommended by teachers for teachers. The oversight of regulatory committees ensure that the content is appropriate and educationally sound. However, no country appears to have satisfactorily organised its recommended digital resources in line with its grade and strand structure – this makes trawling through dozens of websites very time-consuming for the teacher who is trying to address a very specific need. Encouragingly, the audit revealed that Massachusetts[3] has just begun a "Grade by Grade" guide to its available digital resources for mathematics. Finally, Finland is noteworthy for its provision of high quality ICT supports for mathematics teaching of special needs pupils.

[3] http://www.ixl.com/standards/massachusetts/math

Assessment and standards

The assessment and standards strand of this audit best illustrates how Ireland has fallen behind its international partners: with the exception of Singapore, all countries articulate clear expectations for children's mathematical learning at set points in their schooling (see Table 5). This is also true for countries which exploit a state-wide formal mathematics assessment programme (such as Northern Ireland and The Netherlands). Whilst there is obvious variation in the terminology and frameworks used, most countries have opted for "*Can Do*" statements that are built upon clearly identifiable skills, competencies, and in some cases, attitudes. Scotland's framework details skills and competencies written in the first person (e.g. *I have explored…*) – this sets a very striking contrast with other systems who favour a formal teacher imposed-judgement. The Scottish emphasis is clearly an attempt to formalise and strengthen self-assessment capacity.

The "set points" for cataloguing the child's development typically correspond to each grade level, however the *Attainment Target Levels* (used in England and in Wales to an extent) do provide a degree of flexibility to accommodate learners who may be performing above or below the expected norm for their grade (age) level. The *Common Core Standards*[4,] which themselves originated from an imperative to devise shared expectations of pupil progress, present their content in two separate styles: a grade-by-grade approach similar to the Irish documents, but alternatively via detailed descriptions of content in the eleven elementary school mathematical "domains". The later style of presentation is particularly useful for maximising integration within the mathematics curriculum. Finland has chosen to carry out their "assessments" at three specified points in a child's elementary schooling; at the conclusion of Grades 2, 5 & 8. Paradoxically, the previously lauded curricular content flexibility evident in New Zealand does not carry through to its rigid grade-by-grade, strand-by-strand application of its *Maths Standards*. Ontario's Achievement Charts[5] are supplemented by fully expanded user-friendly descriptors (*Limited – Some – Considerable – Thorough*) to allow teachers make more specific judgements about their pupils. Australia[6] uses a more simplistic, yet equally effective, *Above/Below & Satisfactory* rubric. Based on the audit, this paper sees a substantial benefit in exploiting some form of "Expectation Framework" to offset the

[4] http://www.corestandards.org/Math
[5] http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf
[6] http://www.australiancurriculum.edu.au/Mathematics/Achievement-standards

growing school and parental fixation with standardised test scores, and their inadequacy in providing a holistic overview of a child's mathematical development.

7. Examples of children's work

As an aid to teachers in exploiting such "Expectation Frameworks", the provision of work samples is considerable, if not completely widespread. However, there is much variation in their value for two key reasons. Firstly, exemplars of pupils' work are provided that bear no relationships to the framework that is set out by the very same curriculum document (e.g. Northern Ireland). Such exemplars only serve to provide ad-hoc and ultimately inapplicable snatches of insight. Secondly, work samples and judgements are provided but without a commentary piece that allows the teacher explore a rationale, and benchmark their own thought-process against best practice. These detractions do not account for countries that undermine a solid "Expectation Framework" by the complete non-provision of pupil work samples (e.g. Finland and England, as of now). Best practice in this key curriculum component is provided by The Netherlands ("*The Calculator Line"* [7] - an excellent interactive tool that is attractive and beneficial in visually reinforcing the concept of a continuum of improvement and skill acquisition), and by Australia and Ontario whose guidelines and work samples are eminently transferable to the busy classroom setting. More specifically the Canadian exemplars, organised on a per grade basis, demonstrate the value of carefully chosen tasks from a range of strands, demonstrating a realistic range of mastery in the pupil responses annotated, and, providing clear guidance for teachers when applying a succinct "Expectation Framework".

Table 5: Articulation of Learning Expectations & Inclusion of Work Samples

	Inclusion of Expectations	**Expectations expressed via**	**Inclusion of Work Samples**
England	Yes	*Attainment Target Level Descriptors* in the relevant *Programme of Study*	No
Finland	Yes	*Description of Good Performance* for each *Core Content* area	No
Hong Kong S.A.R.	Yes	*Basic Competencies Framework*	No
Massachusetts/Common Core Standards	Yes	*Standards Framework*	Yes
New Zealand	Yes	*Maths Standards*	Yes

[7] http://www.fi.uu.nl/rekenlijn/

	Inclusion of Expectations	**Expectations expressed via**	**Inclusion of Work Samples**
Northern Ireland	Yes	The specified skill of *Using Mathematics*	No
The Netherlands	Yes	*Reference Levels for Mathematics*	Yes
Ontario	Yes	*Achievement Charts for Mathematics*	Yes
Queensland	Yes	*Achievement Standards*	Yes
Scotland	Yes	*Experiences & Outcomes* Framework	No
Singapore	No	n/a	No
Wales	Yes	*Attainment Target Level* Descriptors	Yes

Concluding remarks

This audit has highlighted international trends in curriculum policy across twelve countries, and one further amalgamated framework from the Unites States. It has highlighted similarities and differences between Ireland's current primary mathematics curriculum, and that of many countries that have undertaken recent revisions. It is now timely for Ireland to undertake an extensive review of its mathematics curriculum: a positive starting point is the apparently similar curriculum content of many countries whose students consistently achieve to very high standards on international assessments (Hong Kong, Singapore and Finland). However, content is only one component of modern curricular provision: this paper clearly demonstrates the equal importance of curriculum supports, and assessment and standards mechanisms. On the basis of this audit, both of these components require significant research and development in order to keep pace with international best practice. More specifically, the provision of resources to assist in curriculum implementation in Ireland will have to take account of the digital and new-media age we now live in; the notion that a once-off publication of a "Teachers' Guidelines" document can match the longevity of the curriculum is itself outdated. The ability of web sources to constantly update, expand and virtually "put" us into the classroom of other teachers makes it an obvious conduit for teacher support. The provision of clear and applicable "Expectation Frameworks" illustrated and bolstered by carefully chosen pupil work samples is now an accepted component of primary mathematics curricula internationally – this represents the biggest challenge, and opportunity, for Ireland's policy makers and curriculum designers.

It is also important to acknowledge obvious limitations of the audit; the sample of countries chosen is relatively small and could not possibly take account of "emerging" curricula that are showing significant improvements in TIMSS and other international assessment programmes, albeit if from a low base (such as Slovakia and Portugal). Secondly, the centralised nature (and mandated-curricular dependence) of the Irish system does reveal a unique cultural identity that no other nation fully mirrors in all aspects. Thirdly, although many of the nations studied could be considered at least bilingual, the availability of materials in the English language was not always guaranteed, thus depriving this desktop audit of potentially key data. The Netherlands exemplifies this obstacle. In a related point, restricted access to various on-line resource and "Learning Community" portals (such as in Scotland and Australia) did minimise the scope of available material. Finally, the voice of the teacher, the implementer at the coalface, is absent from the paper. The view of educators in the high-achieving Asian countries, for example, could assist in providing some insight into the considerable and sustained success of seemingly orthodox and common-place syllabi. The success or otherwise of any curriculum, including the applicability of its various components and supports, is determined by the experiences of the teacher in his/her classroom; policy-makers should consider how this vital perspective can be accessed.

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www.corestandards.org/Math/Practice/

Url links to mathematics curricula included in the audit are provided as footnotes throughout the report.